ELEG 404/604 - Digital Image and Audio Signal Processing

Computed Tomography

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X-Ray discovery*

In 1895 Wilhelm Rontgen discovered the X-rays, while working with a cathode ray tube in his laboratory. One of his first experiments was a film of his wife's hand.

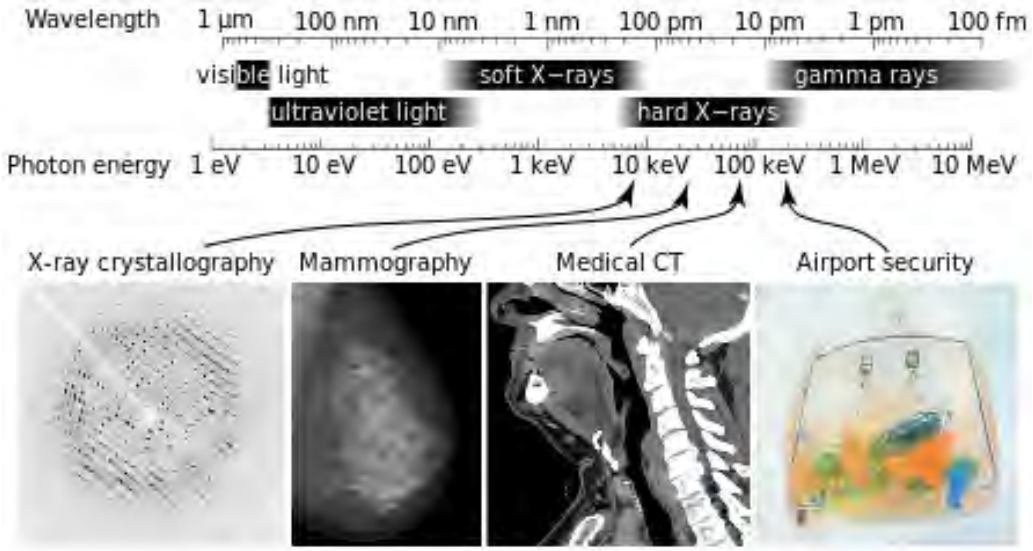
*Based on slides from Mark Mirotznik (ELEG 679)

Shoe Fitting X-Ray Device

Shoe stores in the 1920s until the 1970s installed X-ray fluoroscope machines as a promotion device.

Oak Ridge Associated Universities
Shoe-Fitting Fluoroscope (ca. 1930-1940)

X-Ray Spectrum



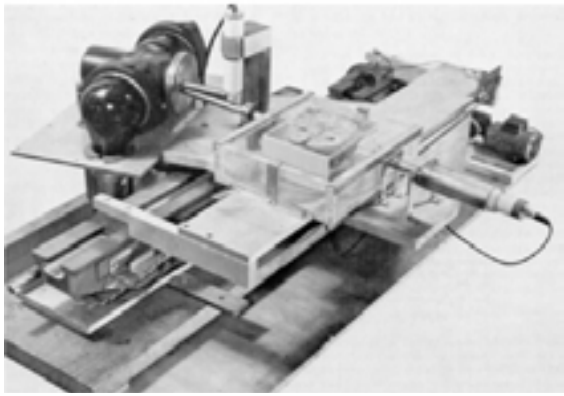
$$E = \hbar \cdot f = \hbar \frac{c}{\lambda}.$$

1 eV is the kinetic energy gained by an electron that is accelerated across a one volt potential.

- **Wavelength**: 0.01 - 10 nm.
- **Frequency**: 30 petahertz ($3x10^{16}$) to 30 exahertz ($3x10^{19}$).
- **Soft X-Rays**: 0.12 to 30 keV.
- **Hard X-Rays**: 30 to 120 keV.

Reconstruction History

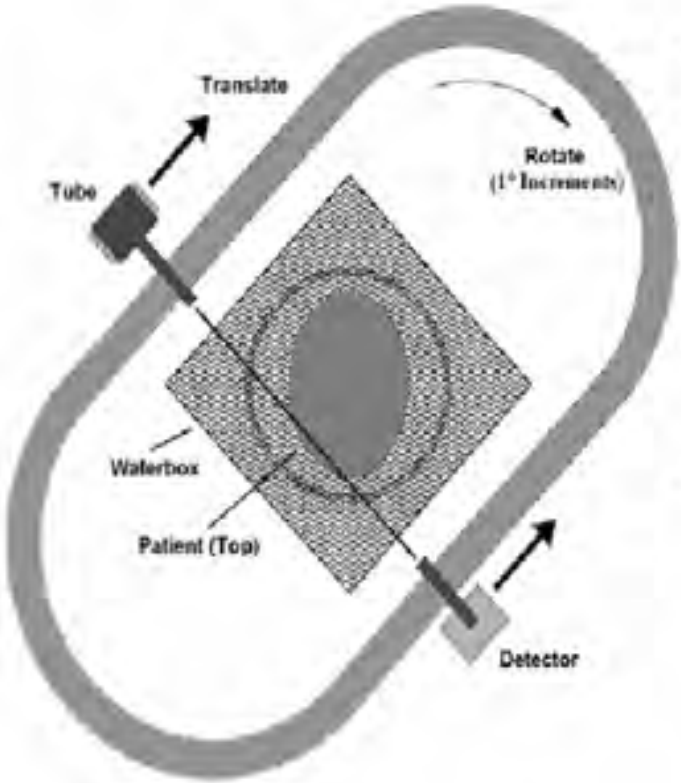
Hounsfield's experimental CT:



- Reconstruction methods based on Radon's work
  - 1917-classic image reconstruction from projections paper
- 1972 - Hounsfield develops the first commercial x-ray CT scanner
- Hounsfield and Cormack receive the 1979 Nobel Prize for their CT contributions
- Classical reconstruction is based on the Radon transform
  - Method known as backprojection
- Alternative approaches
  - Fourier Transform and iterative series-expansion methods
  - Statistical estimation methods
  - Wavelet and other multiresolution methods
  - Sub-Nyquist sampling: Compressed sensing and Partial Fourier Theories

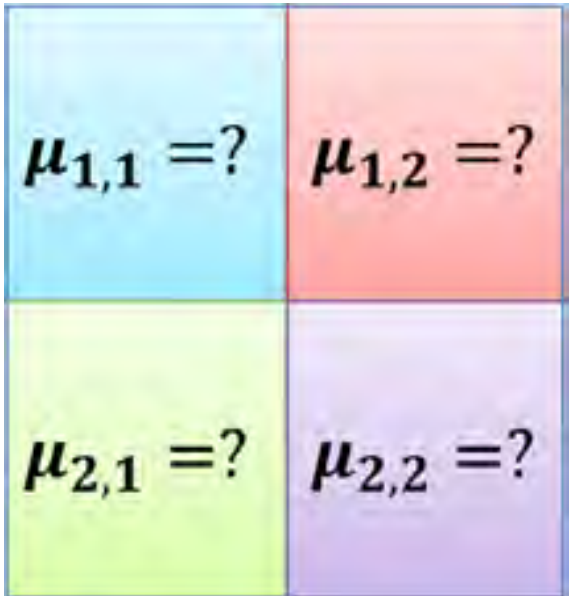
1^{st} Generation CT: Parallel Projections

Hounsfield's Experimental CT



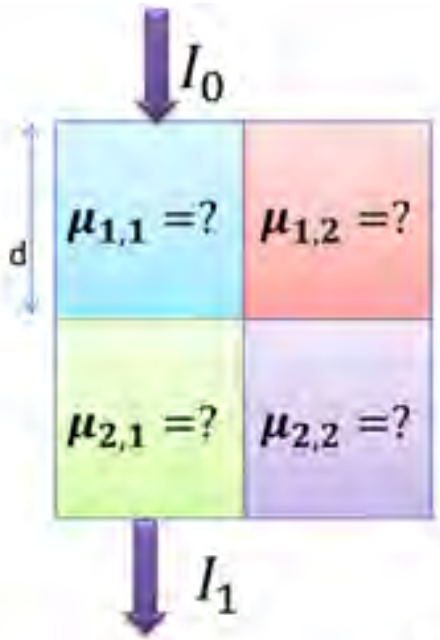
- 1 Beam and 1 Detector
- 160 samples/traverse: 5min
- 1^o increments over 180^o
- 28,800 samples
- Solved simultaneous equations (Fortran): 2.5h
- 160^2 image matrix but reduced to 80^2 for practical clinical use

Example



Suppose an object that has 4 materials arranged in the boxes shown above. How can we find the linear attenuation coefficients?

Image Reconstruction



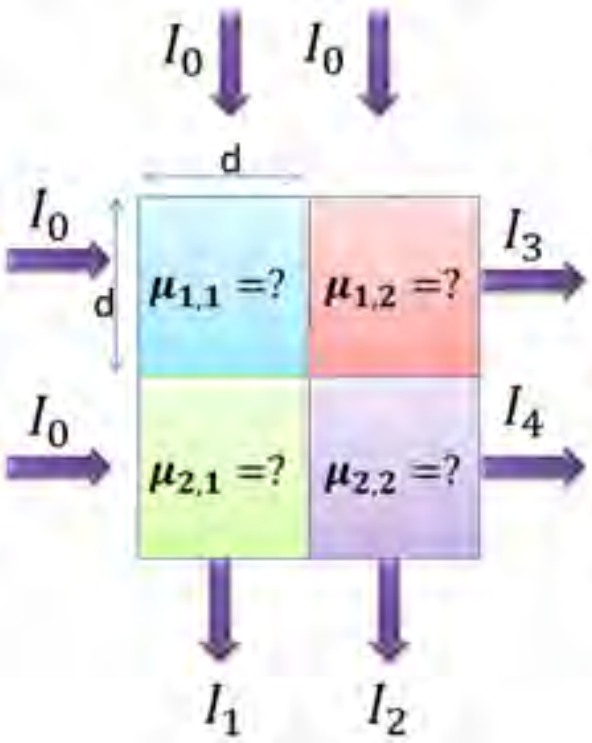
$$I_1 = I_0 e^{-(\mu_{1,1}+\mu_{2,1})d}$$

$$\ln\frac{I_1}{I_0} = -(\mu_{1,1}+\mu_{1,2})d$$

$$(\mu_{1,1}+\mu_{1,2}) = -\frac{1}{d}\ln\frac{I_1}{I_0}$$

Suppose an x-ray of intensity I_0 is passing through the first column of the object, and that I_1 is the intensity measured at the other side.

Image Reconstruction



$$(\mu_{1,1} + \mu_{2,1}) = -\frac{1}{d}\ln\frac{I_1}{I_0}$$

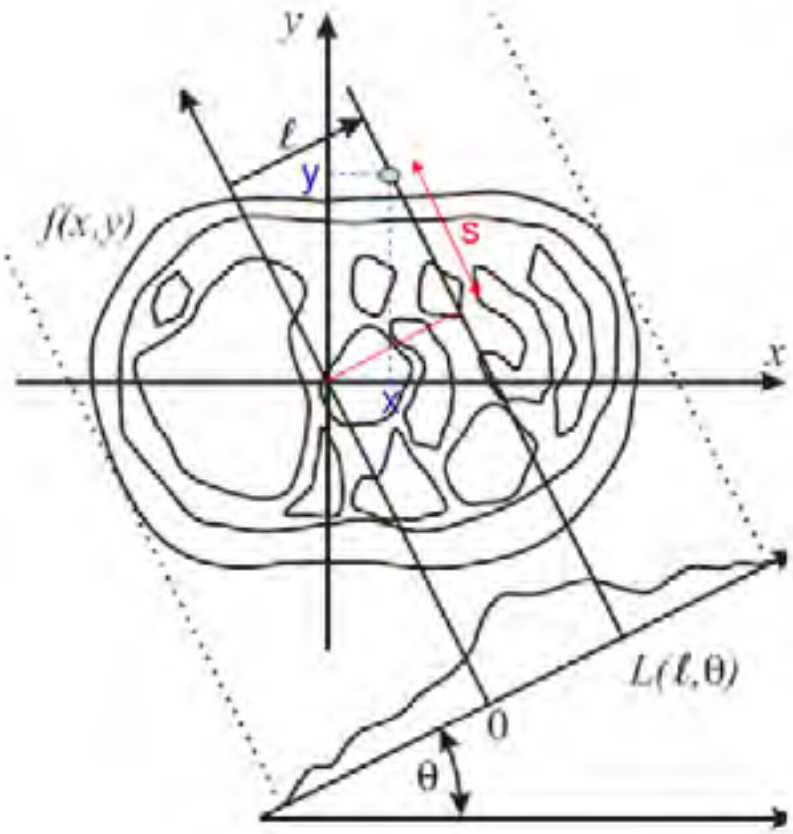
$$(\mu_{1,2} + \mu_{2,2}) = -\frac{1}{d}\ln\frac{I_2}{I_0}$$

$$(\mu_{1,1} + \mu_{1,2}) = -\frac{1}{d}\ln\frac{I_3}{I_0}$$

$$(\mu_{2,1} + \mu_{2,2}) = -\frac{1}{d}\ln\frac{I_4}{I_0}$$

If we repeat the same process for each of the rows and the columns, we obtain the equations necessary to obtain the values of the coefficients.However for bigger systems, the number of equations is not practical for implementation.

Radon Transform

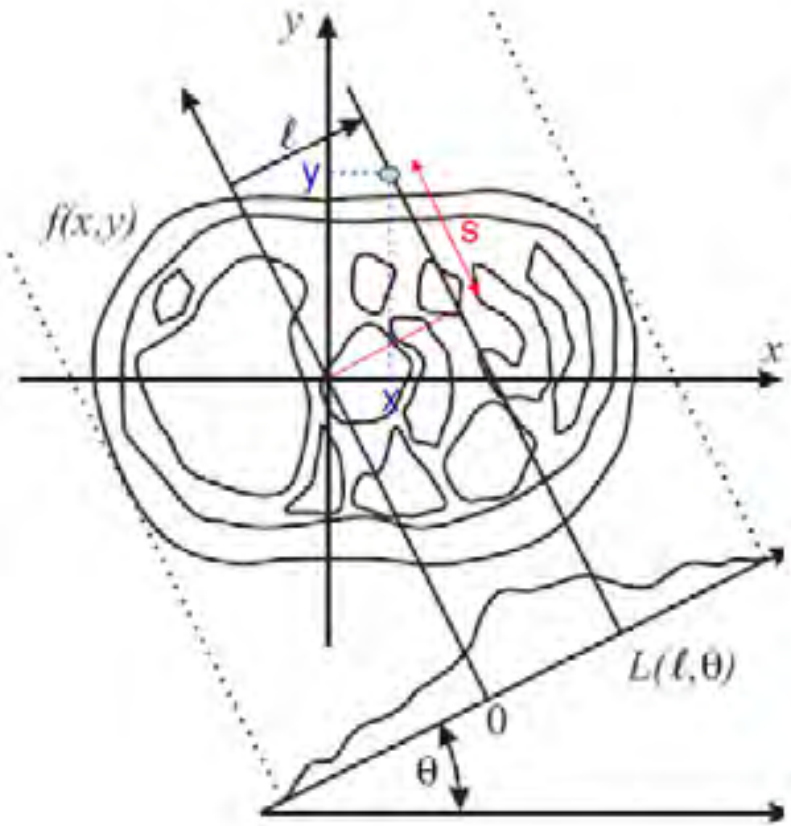


f(x,y) describes our object

We would like to now describe f(x,y) in terms of its projections onto a line L(l,θ).

Here l is a distance along the line L(l,θ) starting from the origin.

Radon Transform



f(x,y) describes our object

We would like to now describe f(x,y) in terms of its projections onto a line L(l,θ).

Here l is a distance along the line L(l,θ) starting from the origin.

Example: If θ=0

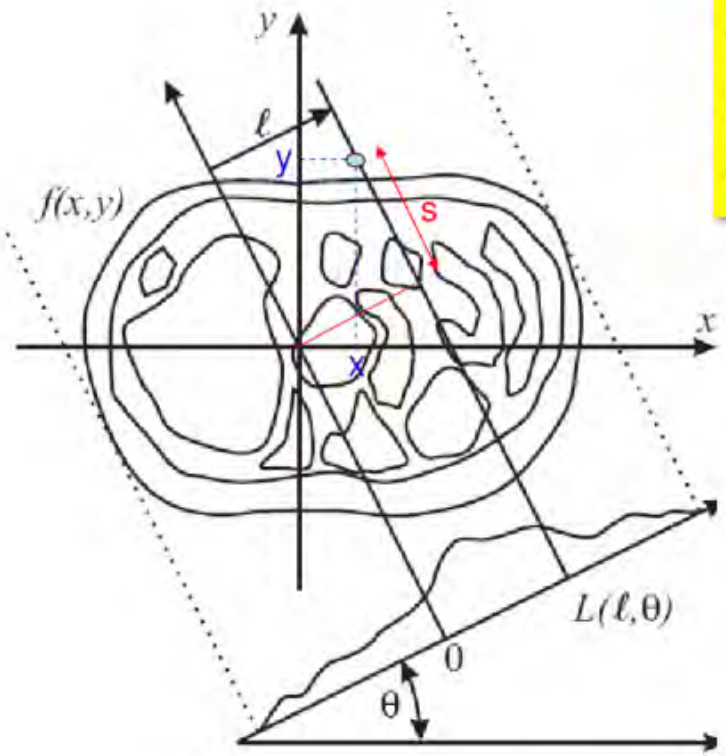
$$g(l,\theta=0)=\int_{-\infty}^{\infty} f(l,y)dy$$

Example: If θ=90°

$$g(l,\theta=90^{\circ})=\int_{-\infty}^{\infty} f(x,l)dx$$

Radon Transform

For Angles different to 0 or 90 degrees.

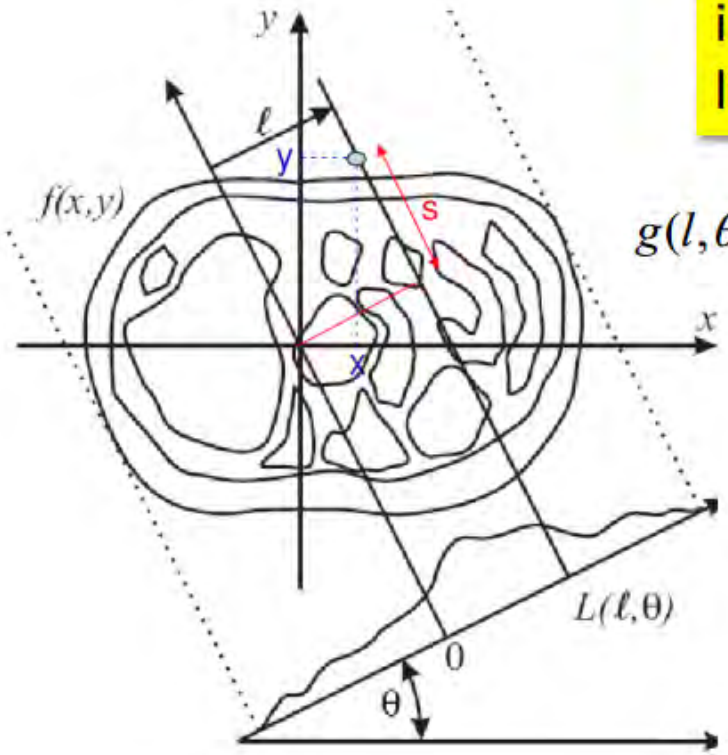


Option 1: We rotate our coordinate system so that l and the projection direction (axis of integration) are horizontal and vertical

$$x(s) = l \cdot \cos(\theta) - s \cdot \sin(\theta)$$
$$y(s) = l \cdot \sin(\theta) + s \cdot \cos(\theta)$$

$$g(l,\theta) = \int_{-\infty}^{\infty} f(x(s), y(s))ds$$

Radon Transform

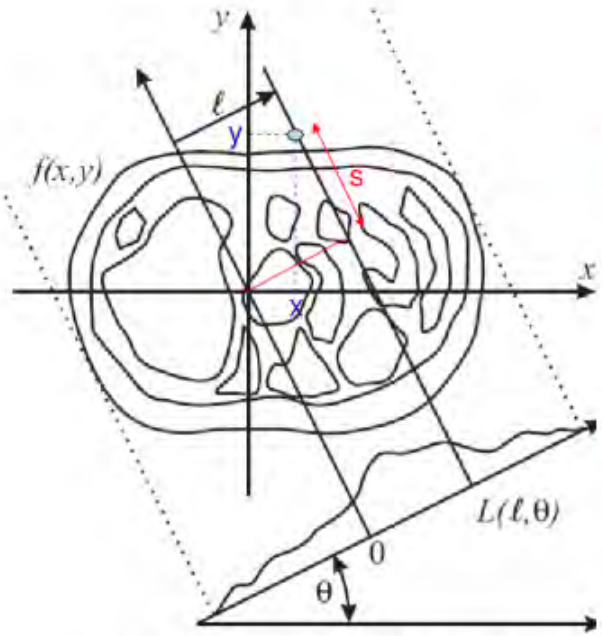


Option 2: Instead of rotating the object and integrating we can integrate the object only along the line (L)

$$g(l,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot\delta(x\cos\theta + y\sin\theta - l)dxdy$$

$g(l,\theta)$ is called the Radon transform

Radon Transform



$$x(s) = l \cdot \cos(\theta) - s \cdot \sin(\theta)$$
$$y(s) = l \cdot \sin(\theta) + s \cdot \cos(\theta)$$

$$g(l,\theta) = \int_{-\infty}^{\infty} f(x(s), y(s))ds$$

How does this apply to xrays?

Recall that $I_d = I_o e^{-\int_0^d \mu(x(s),y(s))ds}$

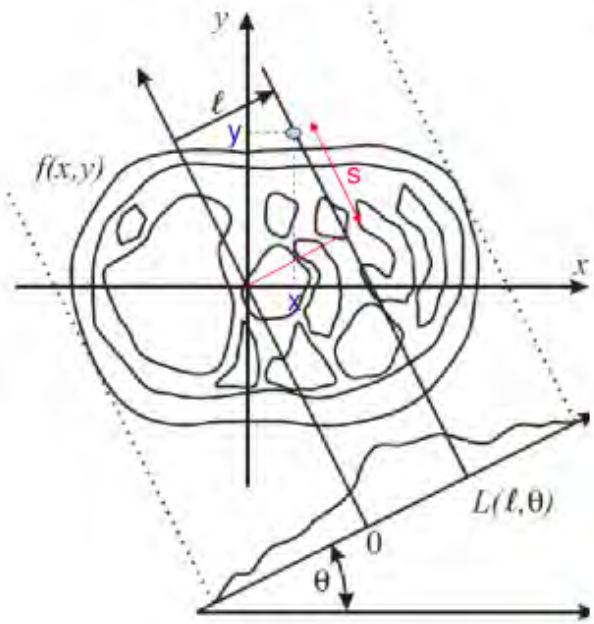
Is the received xray intensity of a beam that is projected through a sample along the line s

$$-\ln(\frac{I_d}{I_o}) = \int_0^d \mu(x(s), y(s))ds$$

Let: $g(l,\theta) = -\ln(\frac{I_d}{I_o})$ and $f(x,y) = \mu(x,y)$

we get the radon transform

Radon Transform



Given $g(l,\theta) = -\ln(\frac{I_d}{I_o})$ and $f(x,y) = \mu(x,y)$

$$x(s) = l \cdot \cos(\theta) - s \cdot \sin(\theta)$$
$$y(s) = l \cdot \sin(\theta) + s \cdot \cos(\theta)$$

$$g(l,\theta) = \int_{-\infty}^{\infty} f(x(s), y(s))ds$$

In CT we measure $g(l,\theta) = -\ln(\frac{I_d}{I_o})$

and need to find $f(x,y) = \mu(x,y)$

using $g(l,\theta) = \int_{-\infty}^{\infty} f(x(s), y(s))ds$

$$x(s) = l \cdot \cos(\theta) - s \cdot \sin(\theta)$$
$$y(s) = l \cdot \sin(\theta) + s \cdot \cos(\theta)$$

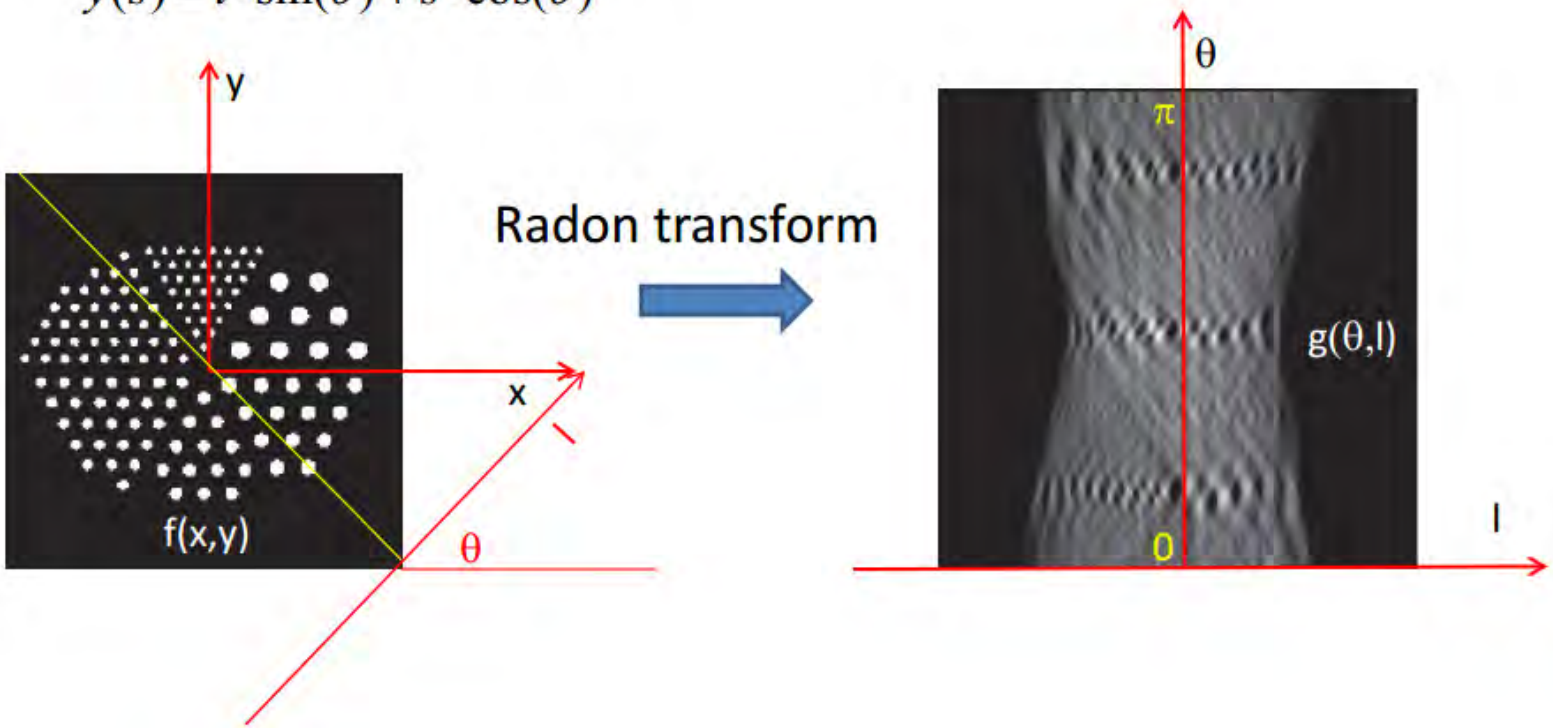
Sinogram

A sinogram is an image of $g(l,\theta)$

$$g(l,\theta) = \int_{-\infty}^{\infty} f(x(s), y(s))ds$$

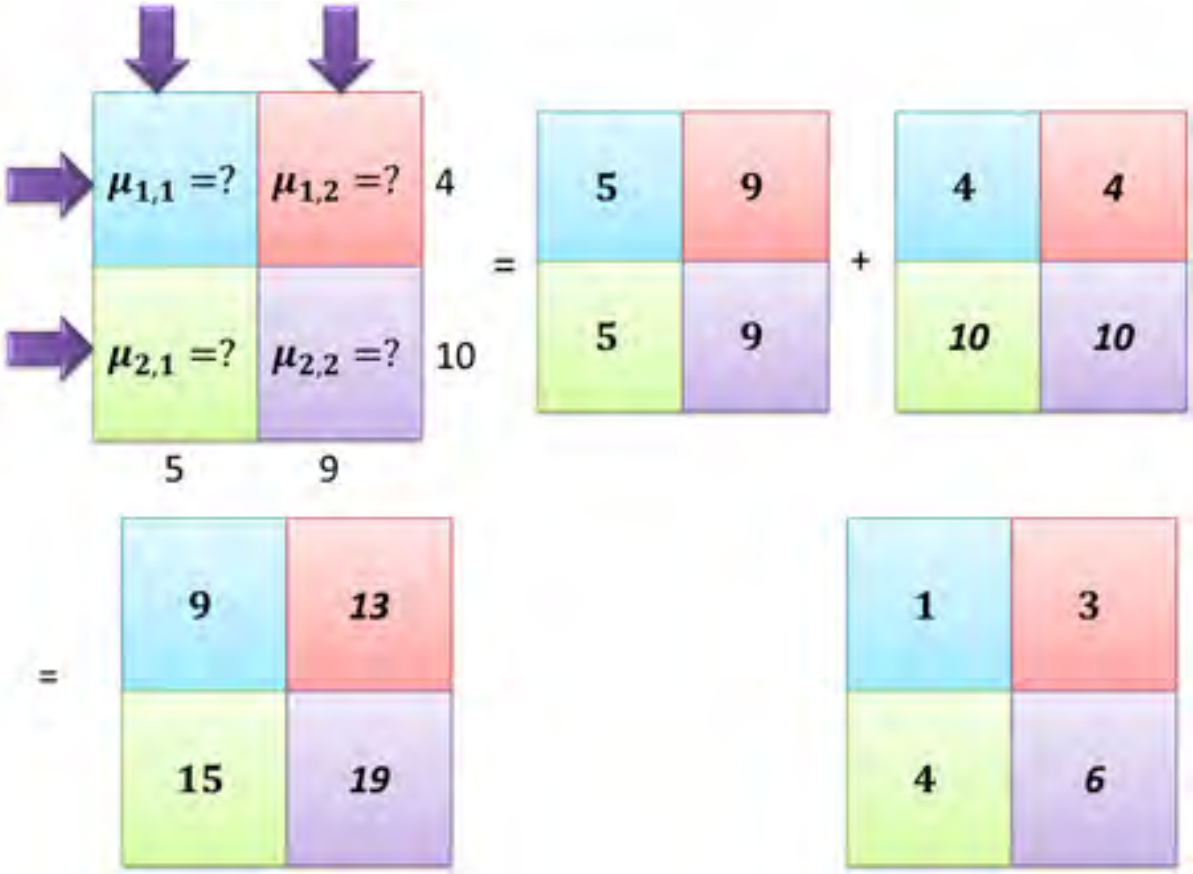
$$x(s) = l \cdot \cos(\theta) - s \cdot \sin(\theta)$$

$$y(s) = l \cdot \sin(\theta) + s \cdot \cos(\theta)$$



Back Projection Example

With the example of the 4 boxes given before, we back project the results obtained. As it can be seen, the right answer is not obtained, however the order of the numbers is the same:



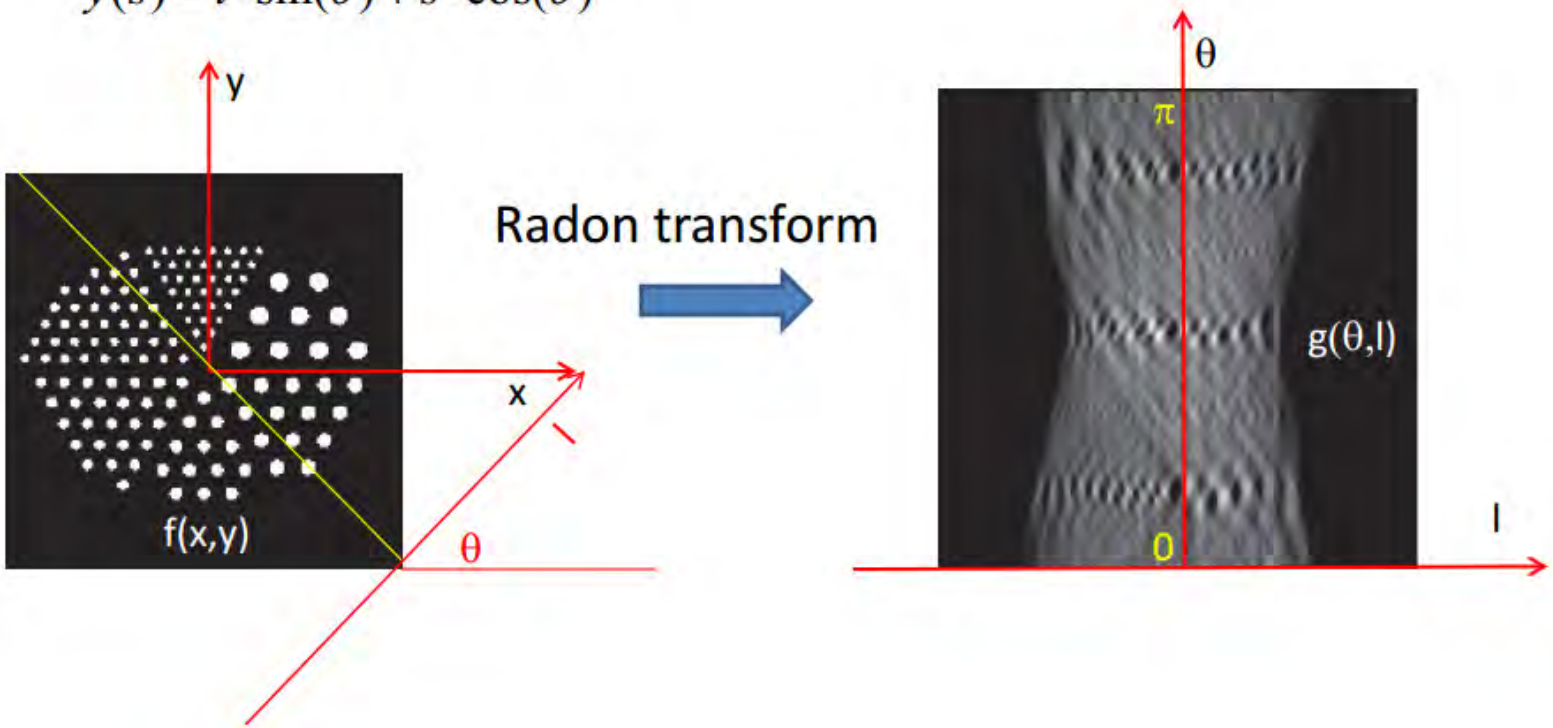
Back Projection Method

A sinogram is an image of $g(l,\theta)$

$$g(l,\theta) = \int_{-\infty}^{\infty} f(x(s), y(s))ds$$

$$x(s) = l \cdot \cos(\theta) - s \cdot \sin(\theta)$$

$$y(s) = l \cdot \sin(\theta) + s \cdot \cos(\theta)$$

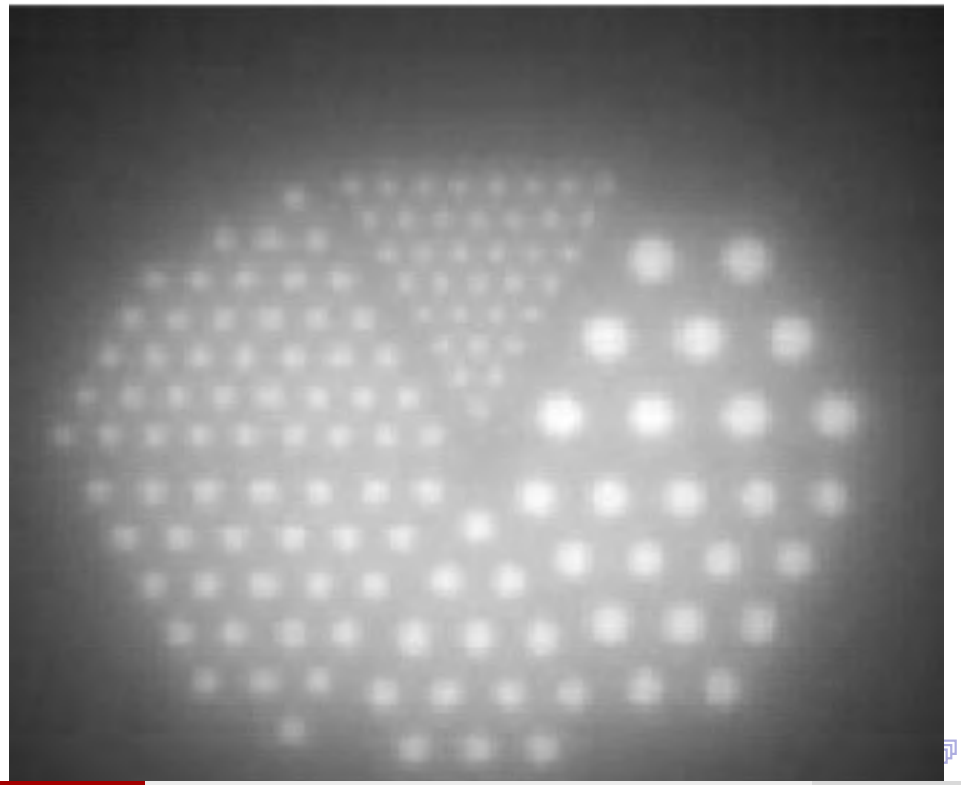


Problems with Back Projection Method

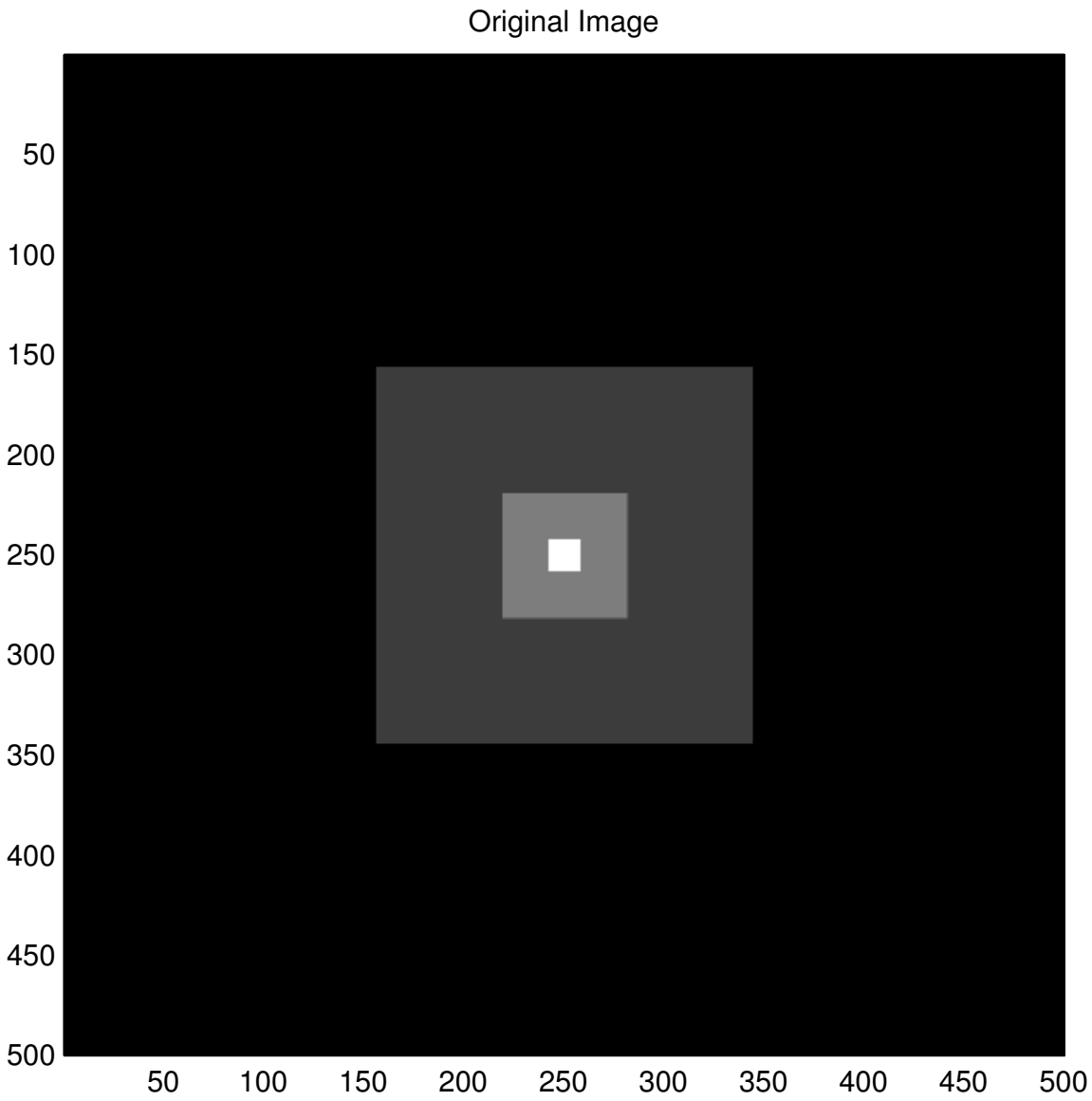
- Bright spots tend to reinforce, which results in a blurry image.
- Problem:

$$f_b(x,y) \neq f(x,y)$$

- Resulting Image:



Back Projection Example



Back Projection Example

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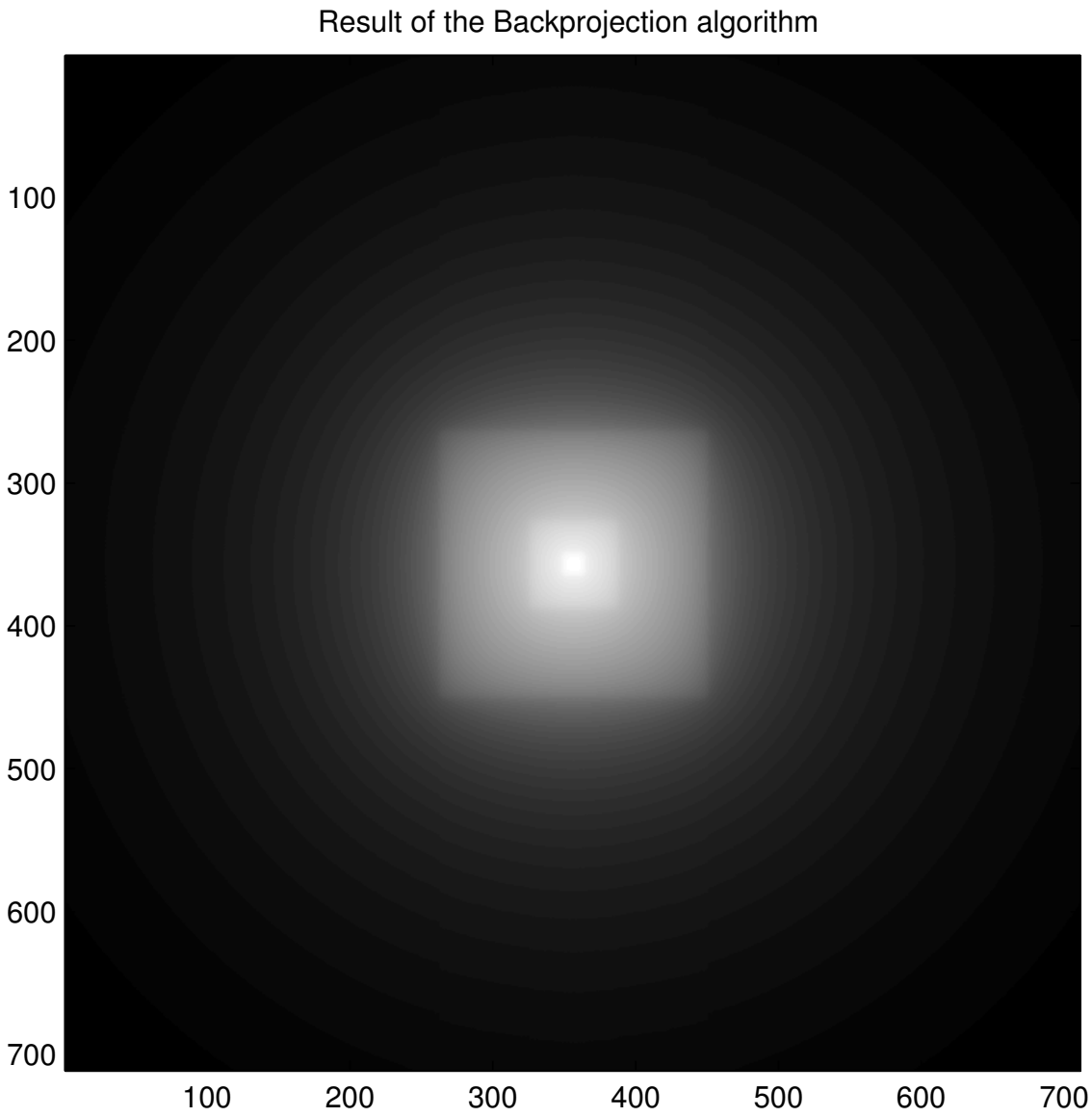
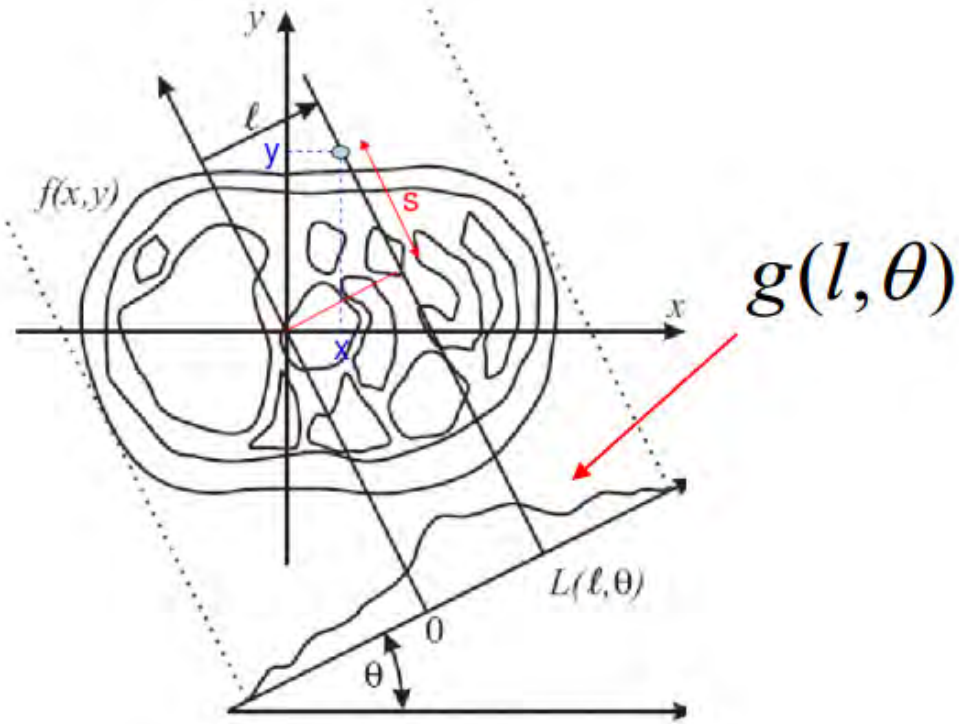
Reconstruction: Back Projection



Projection-Slice Theorem

First take the 1D Fourier transform of a projection $g(l,\theta)$

$$G(\rho,\theta) = \Im_{1D}\{g(l,\theta\} = \int_{-\infty}^{\infty} g(l,\theta)e^{-j2\pi pl}dl$$



Projection-Slice Theorem

From the 1D Fourier transform of a projection $g(l,\theta)$

$$G(\rho,\theta) = \Im_{1D}\{g(l,\theta\} = \int_{-\infty}^{\infty} g(l,\theta)e^{-j2\pi\rho l}\,dl$$

Next we substitute the Radon transform for g(l,θ)

$$g(l,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot\delta(x\cos\theta + y\sin\theta - l)\,dxdy$$

$$G(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot\delta(x\cos\theta + y\sin\theta - l)e^{-j2\pi\rho l}\,dxdydl$$

Next we do a little rearranging

$$G(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\left\{\int_{-\infty}^{\infty}\delta(x\cos\theta + y\sin\theta - l)e^{-j2\pi\rho l}\,dl\right\}dxdy$$

Projection-Slice Theorem

$$G(\rho,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\left\{e^{-j2\pi\rho[x\cos\theta+y\sin\theta]}\right\}dxdy$$

$$G(\rho,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\left\{e^{-j2\pi[x\rho\cos\theta+y\rho\sin\theta]}\right\}dxdy$$

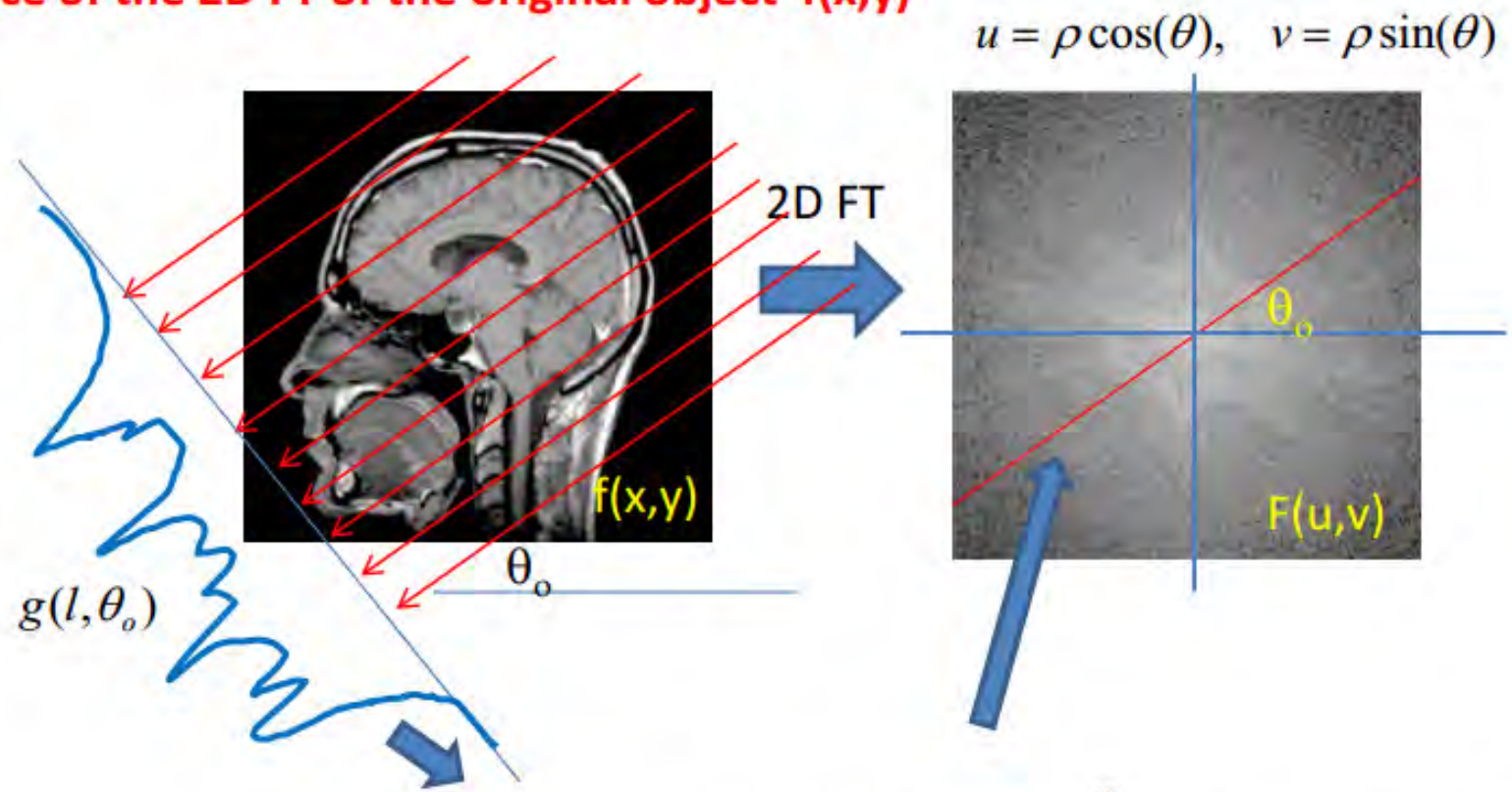
What does this look like?

This looks a lot like $F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi[xu+yv]}dxdy$

with $u=\rho\cos(\theta), \quad v=\rho\sin(\theta)$

Projection-Slice Theorem

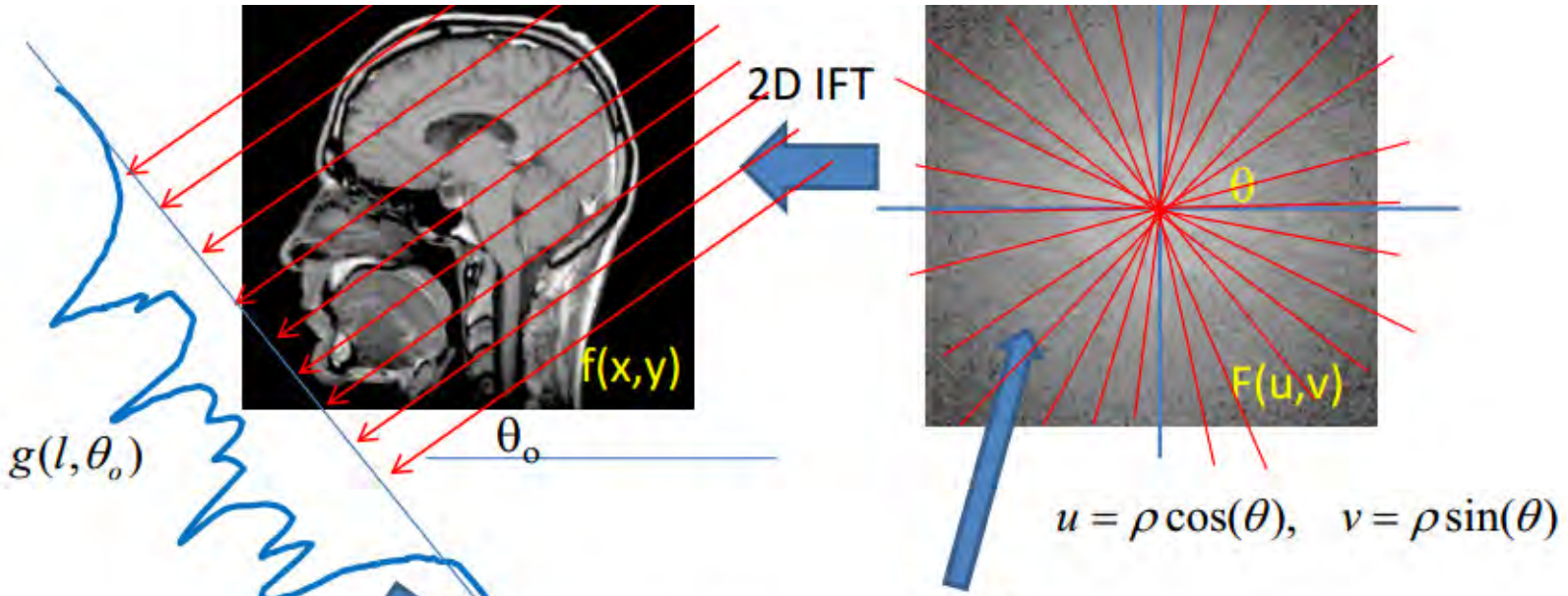
If I take the 1D FT of a projection at an angle θ the result is the same as a slice of the 2D FT of the original object f(x,y)



$$G(\rho,\theta_o) = \Im_{1D}\{g(l,\theta_o)\} = \int_{-\infty}^{\infty} g(l,\theta_o)e^{-j2\pi\rho l}dl$$

$$u = \rho\cos(\theta), \quad v = \rho\sin(\theta)$$

Fourier Reconstruction Method



Take projections at all angles θ.

Take 1D FT of each projection to build F(u,v) one slice at a time.

Take the 2D inverse FT to reconstruct the original object based on F(u,v)

$$f(x,y) = \Im_{2D}^{-1}\{G(\rho,\theta)\}$$

Fourier Reconstruction Method

- The projection slice theorem leads to the following reconstruction method:
  - Take 1D Fourier Transform of each projection to obtain $G(\rho, \theta)$ for all θ.
  - Convert $G(\rho, \theta)$ to Cartesian grid $F(u, v)$.
  - Take inverse 2D Fourier Transform to obtain f(x,y).
- It is not used because it is difficult to interpolate polar data into a Cartesian grid, and the inverse 2D Fourier Transform is time consuming

Filtered Back Projection

Consider the inverse Fourier Transform in 2D:

In polar coordinates the inverse Fourier transform can be written as

$$f(x,y)=\int_0^{2\pi}\int_{-\infty}^{\infty} F(\rho\cos\theta,\rho\sin\theta)e^{j2\pi\rho[x\cos\theta+y\sin\theta]}\rho d\rho d\theta$$

with $u=\rho\cos(\theta), \quad v=\rho\sin(\theta)$

From the projection theorem $G(\rho,\theta)=F(\rho\cos(\theta),\rho\sin(\theta))$

We can write this as $f(x,y)=\int_0^{2\pi}\int_{-\infty}^{\infty} G(\rho,\theta)e^{j2\pi\rho[x\cos\theta+y\sin\theta]}\rho d\rho d\theta$

Filtered Back Projection

Since $g(l,\theta) = g(-l,\theta+\pi)$ you can show

$$f(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} G(\rho,\theta) e^{j2\pi\rho[x\cos\theta + y\sin\theta]} |\rho| d\rho d\theta$$

which can be rewritten as

$$f(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} G(\rho,\theta) e^{j2\pi\rho l} |\rho| d\rho \right]_{l=x\cos(\theta)+y\sin(\theta)} d\theta$$

Filtered Back Projection

- Filter Response.
  - $c(\rho) = |\rho|$.
  - High pass filter.
- $G(\rho, \theta)$ is more densely sampled when ρ is small.
- The ramp filter compensate for the sparser sampling at higher ρ.

Filtered Back Projection Example

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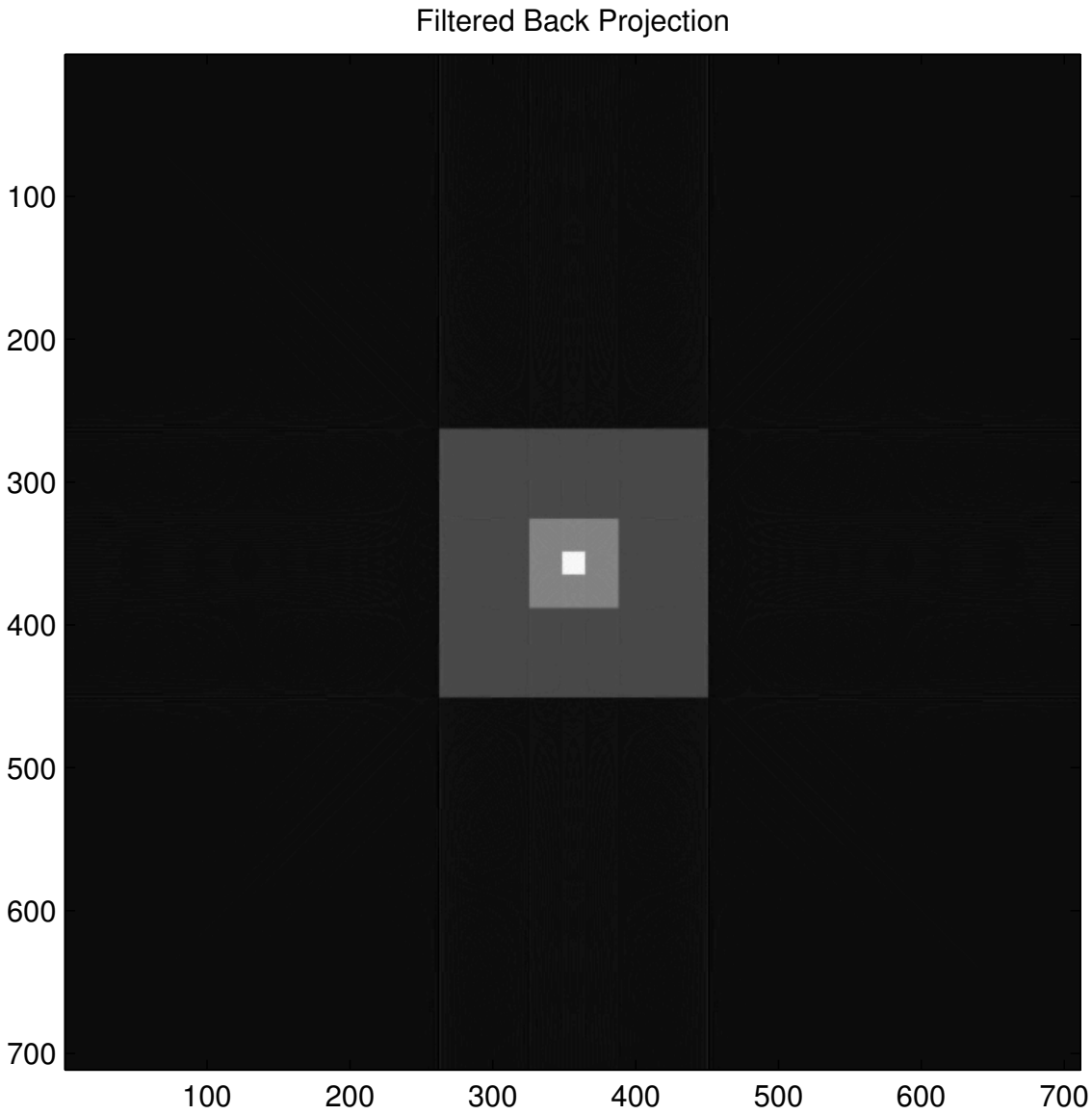
Reconstruction: Filtered Back Projection



Back Projection Method Vs Filtered Back Projection

A. Back Projection

$$b_\theta(x,y) = g(x\cos(\theta) + y\sin(\theta), \theta)$$

$$f_b(x,y) = \int_0^\pi b_\theta(x,y)\, d\theta$$

B. Filtered Back Projection

$$G(\rho,\theta) = \Im_{1D}\{g(l,\theta\} = \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi\rho l}\, dl$$

$$f(x,y) = \int_0^\pi \left[\int_{-\infty}^{\infty} G(\rho,\theta) e^{j2\pi\rho l} |\rho| d\rho \right]_{l=x\cos(\theta)+y\sin(\theta)} d\theta$$

Convolution Back Projection

From the filtered back projection algorithm we get

$$f(x,y)=\int_0^{\pi}\left[\int_{-\infty}^{\infty}G(\rho,\theta)e^{j2\pi\rho l}|\rho|d\rho\right]_{l=x\cos(\theta)+y\sin(\theta)}d\theta$$

It may be easier computationally to compute the inner 1D IFT using a convolution

recall $\Im_{1D}^{-1}\left[F_1(\omega)\cdot F_2(\omega)\right]=f_1(x)*f_2(x)$

$$\Rightarrow\quad f(x,y)=\int_0^{\pi}\left[g(l,\theta)*\Im_{1D}^{-1}(|\rho|)\right]_{l=x\cos(\theta)+y\sin(\theta)}d\theta$$

Convolution Back Projection

$$f(x,y)=\int_0^{\pi}\left[g(l,\theta)*\Im_{1D}^{-1}(|\rho|)\right]_{l=x\cos(\theta)+y\sin(\theta)}$$

Let

$$c(l)=\Im_{1D}^{-1}(|\rho|)$$

$$f(x,y)=\int_0^{\pi}\left[g(l,\theta)*c(l)\right]_{l=x\cos(\theta)+y\sin(\theta)}d\theta$$

The problem is $c(l)=\Im_{1D}^{-1}(|\rho|)=\int_{-\infty}^{\infty}|\rho|e^{j2\pi\rho l}d\rho$ does not exist

Convolution Back Projection

The solution $\tilde{c}(l) = \Im_{1D}^{-1}(|\rho| \cdot W(\rho)) = \int_{-\infty}^{\infty} |\rho| \cdot W(\rho) e^{j2\pi\rho l} d\rho$

where $W(\rho)$ is called a weighting function

$$f(x,y) = \int_{0}^{\pi} [g(l,\theta) * \tilde{c}(l)] d\theta$$

$$\tilde{c}(l) = \Im_{1D}^{-1}(|\rho| \cdot W(\rho)) = \int_{-\infty}^{\infty} |\rho| \cdot W(\rho) e^{j2\pi\rho l} d\rho$$

Convolution Back Projection

$$f(x,y) = \int_0^{\pi} [g(l,\theta) * \tilde{c}(l)] d\theta$$

$$\tilde{c}(l) = \Im_{1D}^{-1}(|\rho| \cdot W(\rho)) = \int_{-\infty}^{\infty} |\rho| \cdot W(\rho) e^{j2\pi\rho l} d\rho$$

Common windows

- Hamming window
- Lanczos window (Sinc function)
- Simple rectangular window
- Ram-Lak window
- Kaiser window
- Shepp-Logan window

Convolution Back Projection Example

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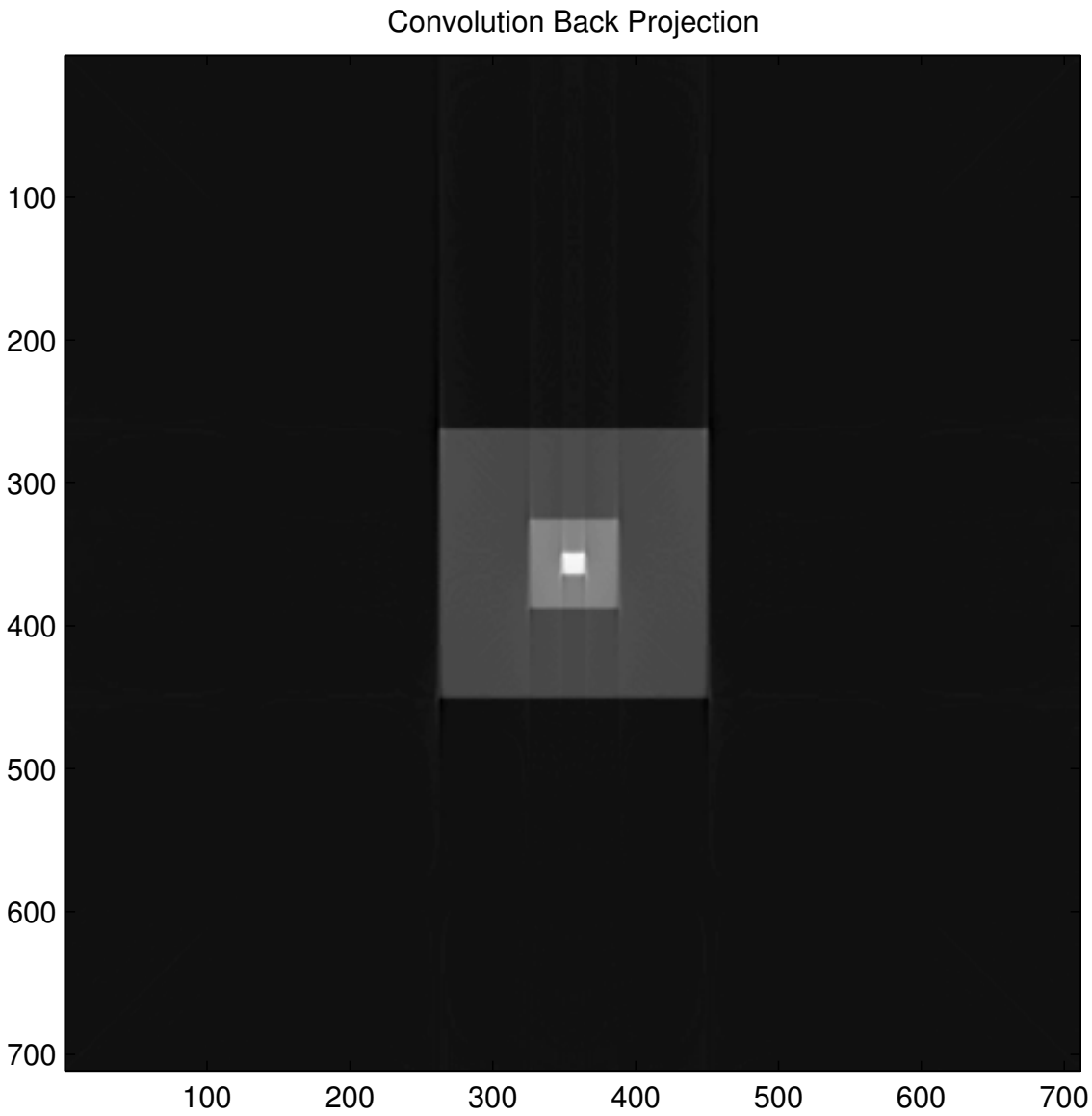
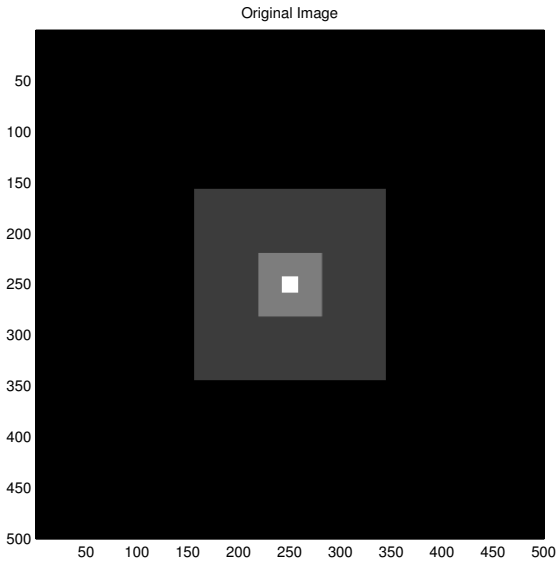
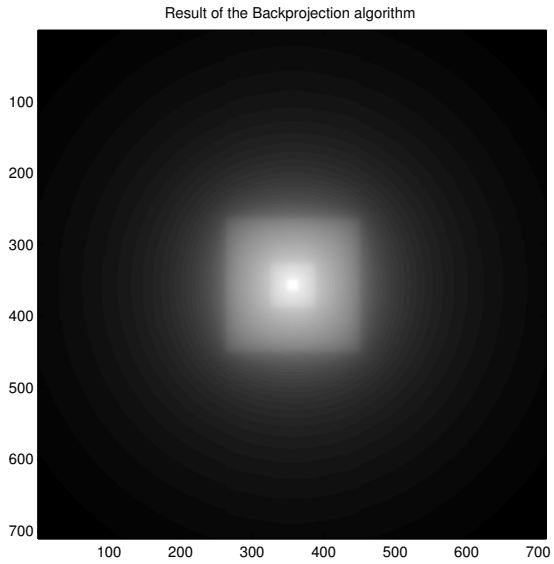
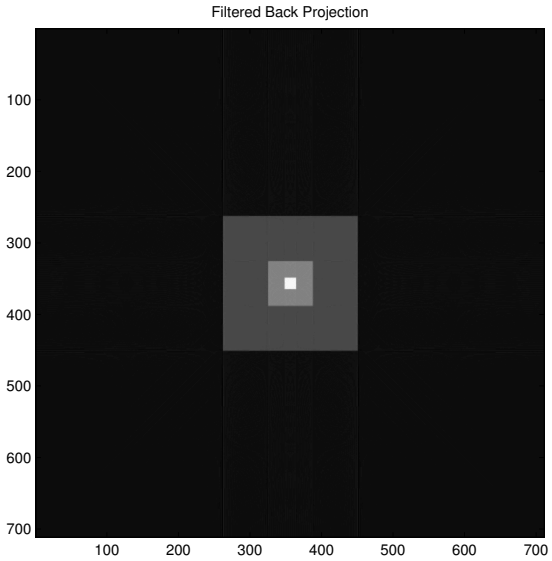
Reconstruction: Convolution Back Projection

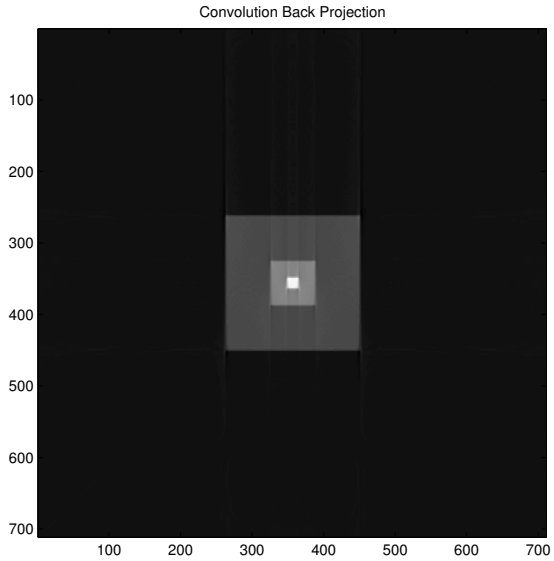


Reconstructions Comparison



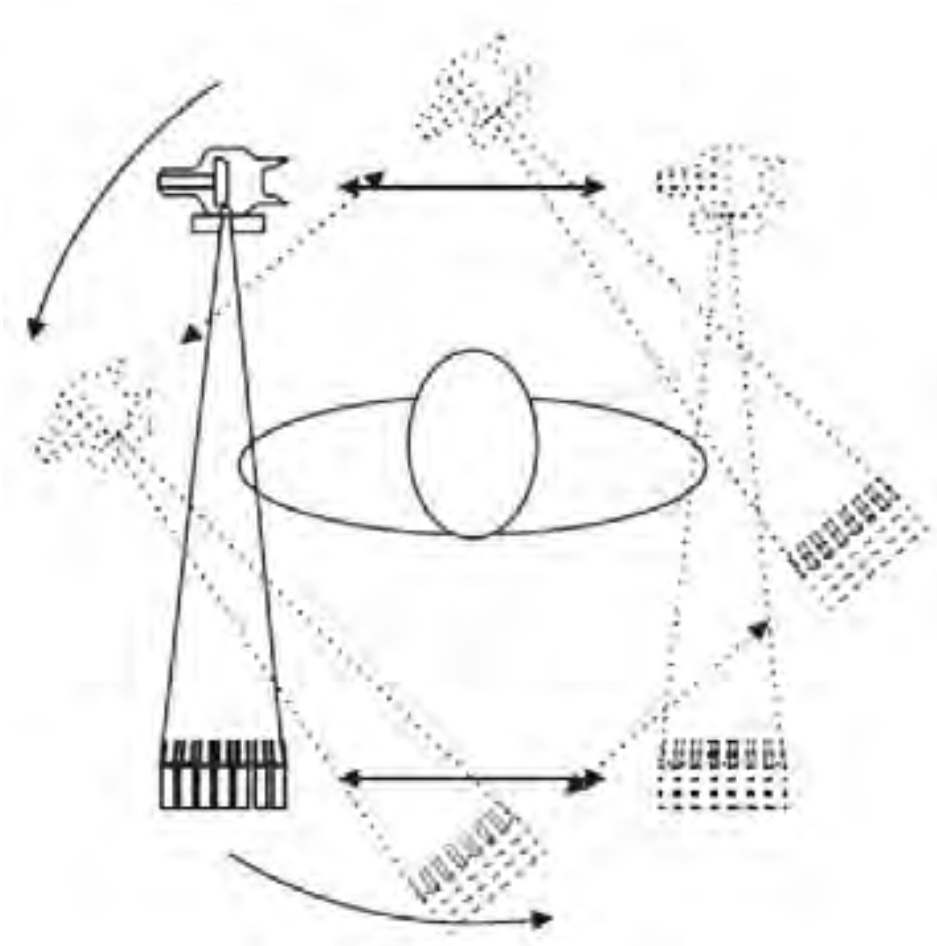




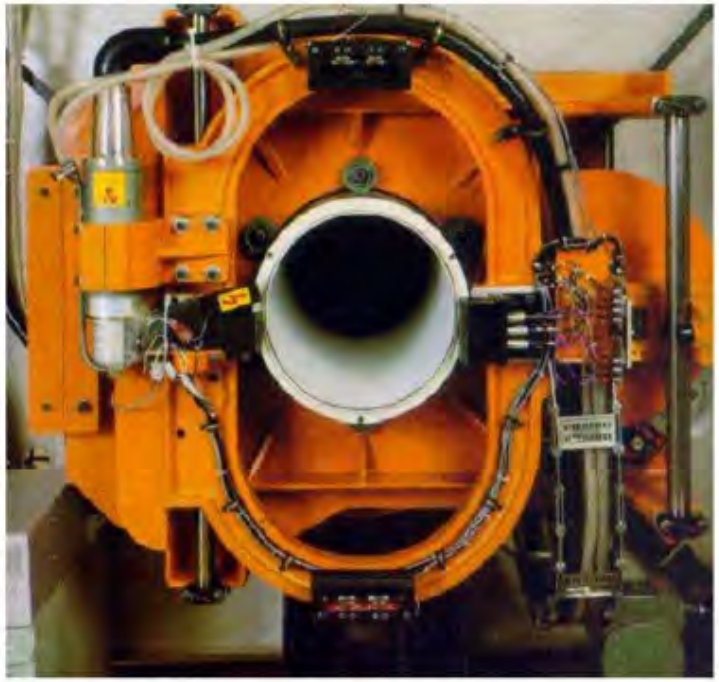


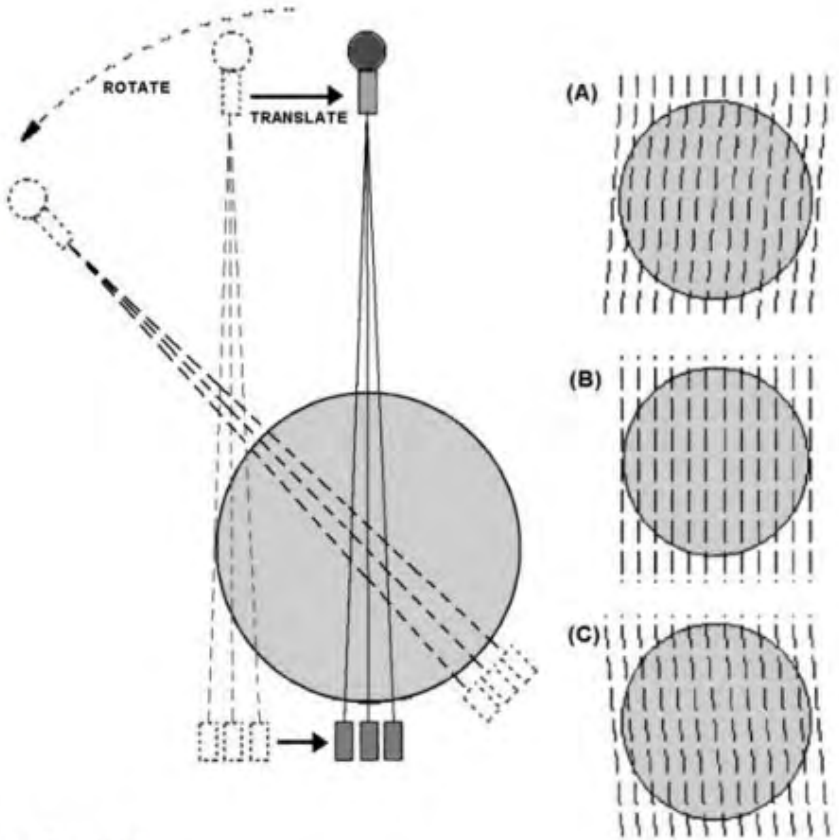
2^{nd} Generation

- Incorporated linear array of 30 detectors
- More data acquired to improve image quality
- Shortest scan time was 18 seconds/slice
- Narrow fan beam allows more scattered radiation to be detected



2^{nd} Generation

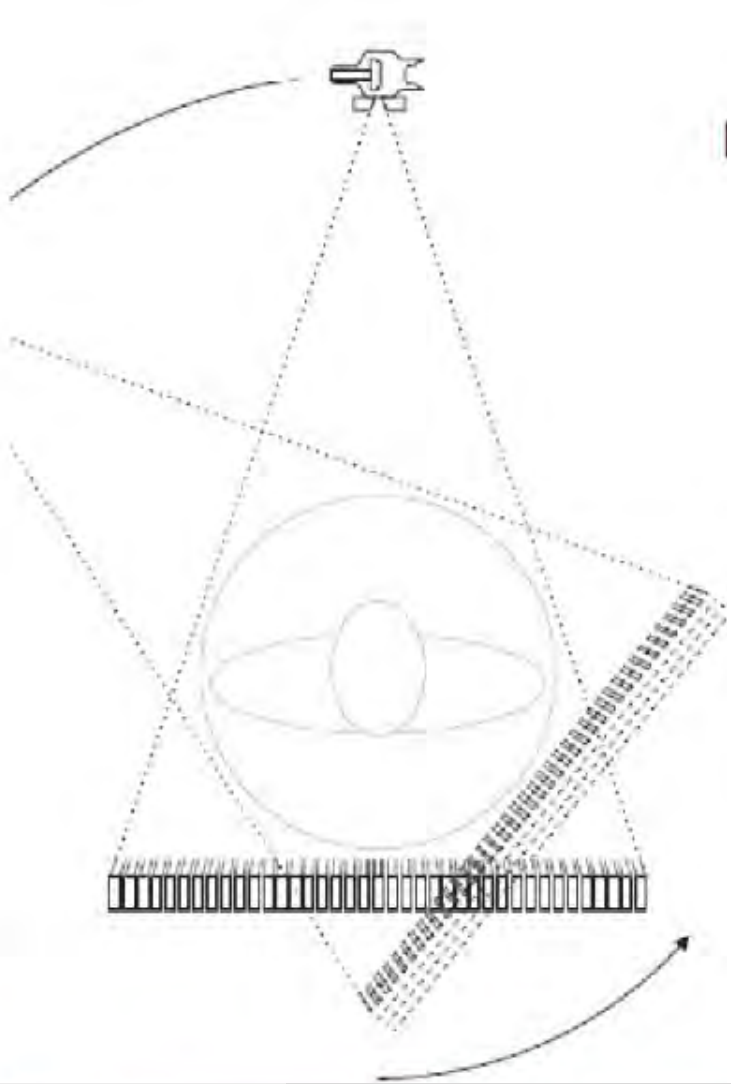




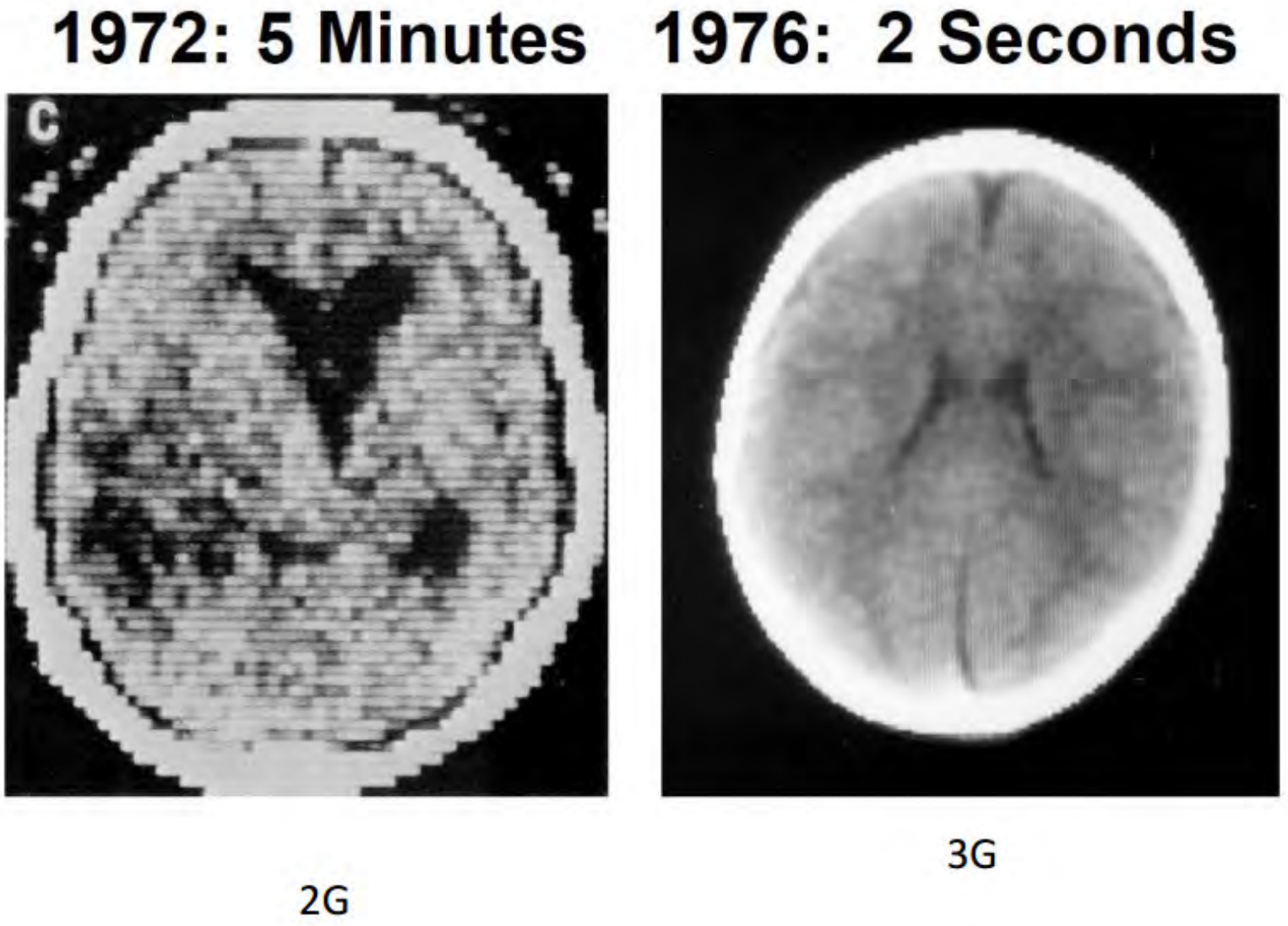
- Multiple detectors
- Still translate-rotate
  - –1 view acquired per detector (~1° apart)
  - – angular increment increased by using more detectors

3rd Generation

- Number of detectors increased substantially (more than 800 detectors)
- Angle of fan beam increased to conver entire patient (no need for translational motion)
- Mechanically joined x-ray tube and detector array rotate together
- Newer systems have scan times of $1/2$ second

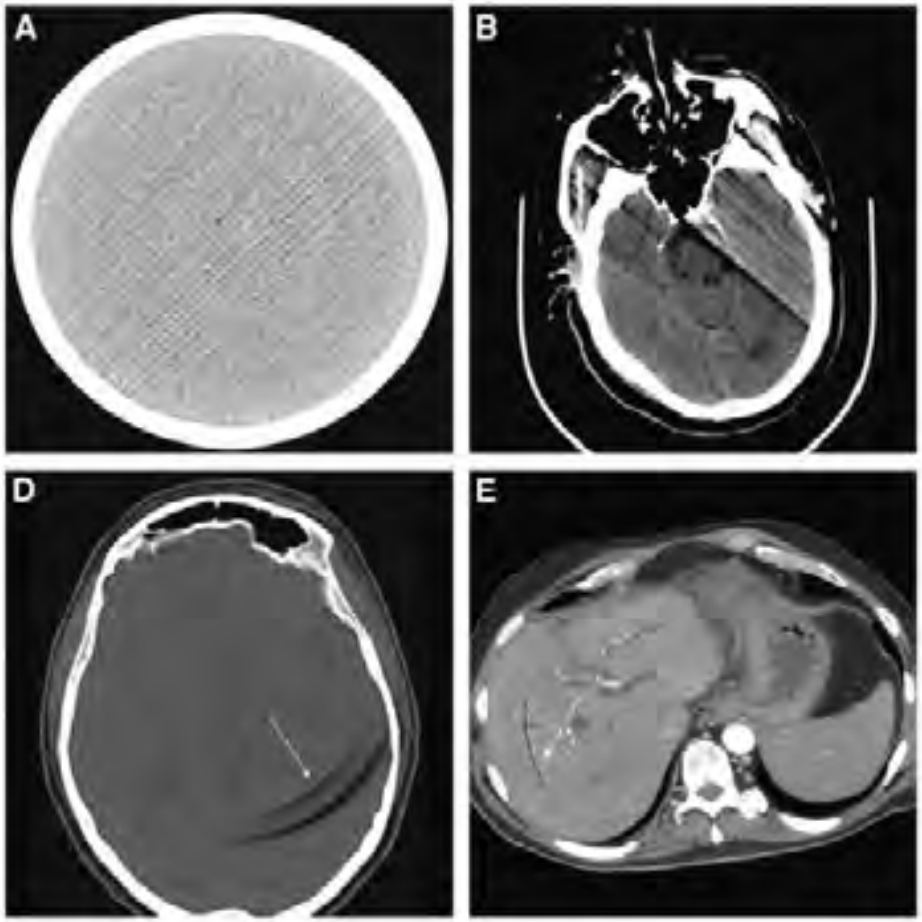


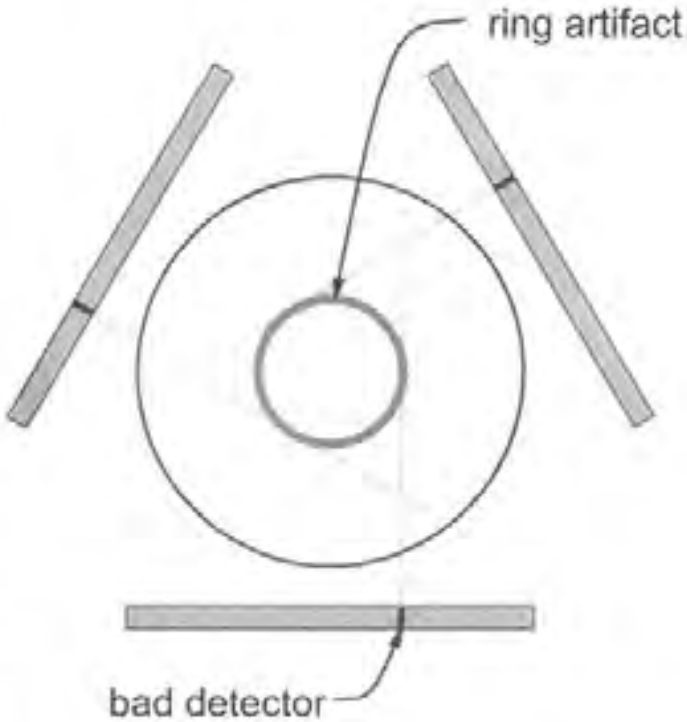
2^{nd} and 3^{rd} Generation Reconstructions



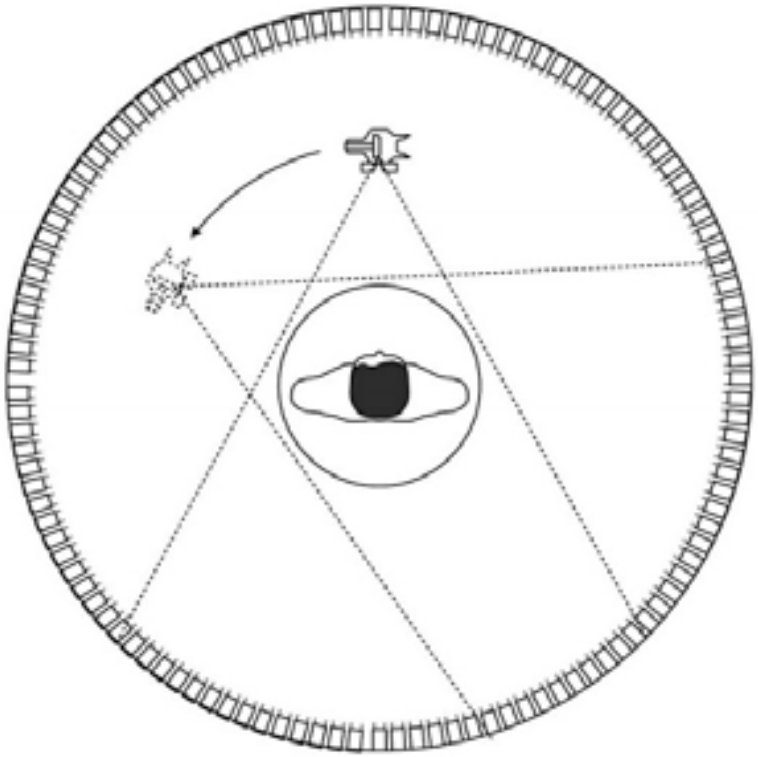
3^{rd} Generation Artifacts

Ring Artifacts





4^{rth} Generation



Designed to overcome the problem of artifacts. Stationary ring of about 4800 detectors