Appendix B

Effective Length of the Infinite Impulse Response

As a measure for the length of the impulse response of a recursive filter, we define the *effective length* N_P as the number of impulse response samples $h(n)$ which bring about $P\%$ or more of its total energy, i.e.,

$$E_{\mathrm{P}} = \sum_{n=0}^{N_P} h^2(n) \geq \frac{P}{100} E \tag{B.1}$$

where n is the discrete time index and E is the total energy of the impulse response defined as

$$E = \sum_{n=0}^{\infty} h^2(n) = \frac{1}{\pi} \int_0^{\pi} \left| H(e^{j\omega}) \right|^2 d\omega \tag{B.2}$$

and $H(e^{j\omega})$ is the frequency response of the system with the impulse response $h(n)$. The length N_P is the value of time index n in (B.1) at which $P\%$ of the energy of the impulse response has arrived. The choice of the value for P depends on the application.

In the following we derive closed-form expressions for the $P\%$ energy length of the first and second-order all-pole filters. The result for the second-order case is particularly interesting because higher order filters are often realized as a cascade or parallel connection of second-order sections.

First-Order All-Pole Filter

The transfer function of the first-order all-pole filter is

$$H(z) = \frac{1}{1 + a_1 z^{-1}} \tag{B.3}$$

where a_1 is real-valued and $|a_1| < 1$. The impulse response of this filter is $(-a_1)^n$ for $n \geq 0$ and 0 for $n < 0$.

$$h(n) = \begin{cases} 0 & \text{for } n < 0 \\ (-a_1)^n & \text{for } n \geq 0 \end{cases} \tag{B.4}$$

The square sum (B.4) can be expressed in closed form as

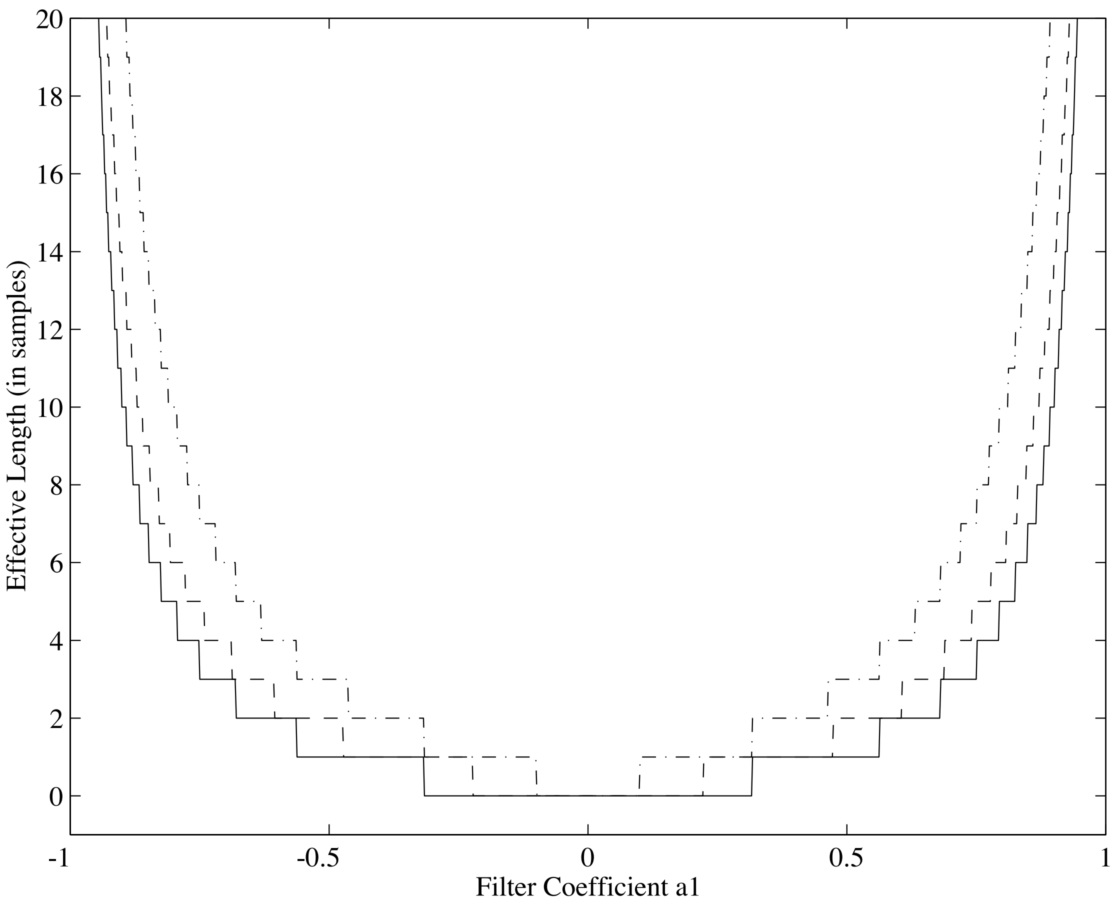

Fig. B.1 Effective length N_P (in samples) of the first-order all-pole filter as a function of the filter coefficient a_1. The curves are for the values P = 90% (solid line), P = 95% (dashed line), and P = 99% (dotted line).

$$E_P = \sum_{n=0}^{N_P} h^2(n) = \sum_{n=0}^{N_P} \left(a_1^2\right)^n = \frac{1-\left(a_1^2\right)^{N_P+1}}{1-a_1^2} \tag{B.5}$$

and the total energy is obtained as a limit as $N_P \to \infty$ to be

$$E = \frac{1}{1-a_1^2} \tag{B.6}$$

For a given P, N_P can be solved from (B.2) by substituting the equations (B.5) and (B.6), that is

$$N_P = \left\lceil \frac{\log(1-P/100)}{\log(a_1^2)} - 1 \right\rceil = \left\lfloor \frac{\log(1-P/100)}{\log(a_1^2)} \right\rfloor \tag{B.7}$$

where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the ceiling and floor operations, which correspond to rounding upwards and downwards. Note that a practical value for N_P must be an integer. The base of the logarithm in (B.7) is arbitrary, e.g., 10. Figure B.1 shows the effective length N_P of the one-pole filter as a function of the filter coefficient a_1 for P = 90%, 95%, and 99%.

Second-Order All-Pole Filter

The transfer function of the second-order purely recursive filter can be written as

$$H(z) = \frac{1}{1 - 2r\cos(\theta)z^{-1} + r^2 z^{-2}} \tag{B.8}$$

The poles of this filter are at $z = re^{\pm j\theta}$, and for stability it is required that $|r| < 1$. The corresponding impulse response can be expressed as

$$h(n) = \begin{cases} 0, & n < 0 \\ \dfrac{r^n}{\sin\theta}\sin[(n+1)\theta], & n \ge 0 \end{cases} \tag{B.9}$$

The total energy of $h(n)$ can be given by (Jury, 1964)

$$E = \frac{1}{\pi}\int_0^{\pi} \left|H(e^{j\omega})\right|^2 d\omega = \frac{1 + r^2}{\left(1 - r^2\right)\left(1 + r^4 - 2r^2\cos 2\theta\right)} \tag{B.10}$$

and the cumulative energy E_P by

$$E_P = \sum_{n=0}^{N_P} h^2(n) = E - E_{100-P} \tag{B.11}$$

where

$$E_{100-P} = \frac{1}{\sin^2\theta}\sum_{n=N_P+1}^{\infty} r^{2n}\sin^2[(n+1)\theta] \le \frac{1}{\sin^2\theta}\sum_{n=N_P+1}^{\infty} r^{2n} = \frac{r^{2(N_P+1)}}{(1-r^2)\sin^2\theta} \tag{B.12}$$

Solving for N_P gives an upper bound as

$$N_P \le N_{P,sim} = \left\lfloor \frac{\log\left[(1-r^2)E_{100-P}\sin^2(\theta)\right]}{\log(r^2)} - 1 \right\rfloor \tag{B.13}$$

Equation (B.12) overestimates the residual energy E_{100-P} and consequently the estimate for E_P (B.11) becomes too small. This is because it has been assumed that $\sin^2[(n+1)\theta] \le 1$. Thus, Eq. (B.13) gives a value too large for N_P.

The estimate in (B.13) for N_P of the second-order recursive filter can be improved by using $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ and by writing (B.12) in the form $E_{100-P} = G_{P,1}^{\copyright} - G_{P,2}^{\copyright}$ with

$$G_{P,1}^{\copyright} = \frac{r^{2(N_P+1)}}{2(1-r^2)\sin^2(\theta)} \tag{B.14}$$

and

$$G_{P,2}^{\copyright} = \frac{r^{2(N_P+1)}}{2\sin^2(\theta)}\sum_{n=0}^{\infty} r^{2n}\cos(2n\theta + \theta_P) \tag{B.15}$$

where $\theta_P = 2(N_P + 2)\theta$. Using $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and the following sin formulas (from (Gradshteyn and Ryzhik, 1964), Eq. 1.447)

$$\sum_{k=0}^{\infty} p^n \sin(k\alpha) = \frac{p \sin\alpha}{1 - 2p\cos(\alpha) + p^2} \tag{B.16}$$

$$\sum_{k=0}^{\infty} p^k \cos(k\alpha) = \frac{1 - p\cos\alpha}{1 - 2p\cos(\alpha) + p^2} \tag{B.17}$$

we obtain (B.15) in the form

$$G^{©}_{P,2} = \left[\frac{r^{2(N_P+1)}}{2\sin^2(\theta)}\right] \frac{F(\theta_P)}{1 - 2r^2\cos(2\theta) + r^4} \tag{B.18}$$

with

$$F(\theta_P) = \cos(\theta_P)\left[1 - r^2\cos(2\theta)\right] - \sin(\theta_P) r^2 \sin(2\theta) \tag{B.19}$$

It is seen that $F(\theta_P)$ depends on $\theta_P = 2(N_P + 2)\theta$ in a random-like manner. However, we can determine bounds for $F(\theta_P)$ and thus for N_P by considering θ_P as a continuous variable in the range [0, 2π] and finding the minimum and maximum of $F(\theta_P)$. The locations of these extrema are

$$\theta_{P1} = \arctan\left[\frac{r^2\cos(2\theta) - 1}{r^2\sin(2\theta)}\right] \quad \text{and} \quad \theta_{P2} = \theta_{P1} + \pi \tag{B.20}$$

from which one yields the maximum $F_{\max}$ and the other the minimum $F_{\min}$. Using (B.14) and (B.17), the upper bound for N_P can be solved as

$$N_{\text{P, max}} = \left\lfloor \frac{\log\left[2E_{100-P}\sin^2(\theta)\right] - \log(G_{\max})}{\log(r^2)} \right\rfloor \tag{B.21}$$

with

$$G_{\max} = \frac{1}{1 - r^2} - \frac{F_{\max}}{1 - 2r^2\cos(2\theta) + r^4} \tag{B.22}$$

In a similar way, the lower bound is obtained as

$$N_{P,min} = \left\lfloor \frac{\log\left[2E_{100-P}\sin^2(\theta)\right] - \log(G_{min})}{\log(r^2)} - 1 \right\rfloor \tag{B.23}$$

where $G_{\min}$ is obtained from (B.22) by replacing $F_{\max}$ by $F_{\min}$.

Figure B.2 illustrates P% energy length N_P (P = 90) of the second-order all-pole filter as a function of θ (r = 0.9). The simple approximation given by (B.13) together with the minimum and maximum estimates (B.21) and (B.23) are presented. For comparison, Fig. B.2 also shows the result of a numerical simulation.

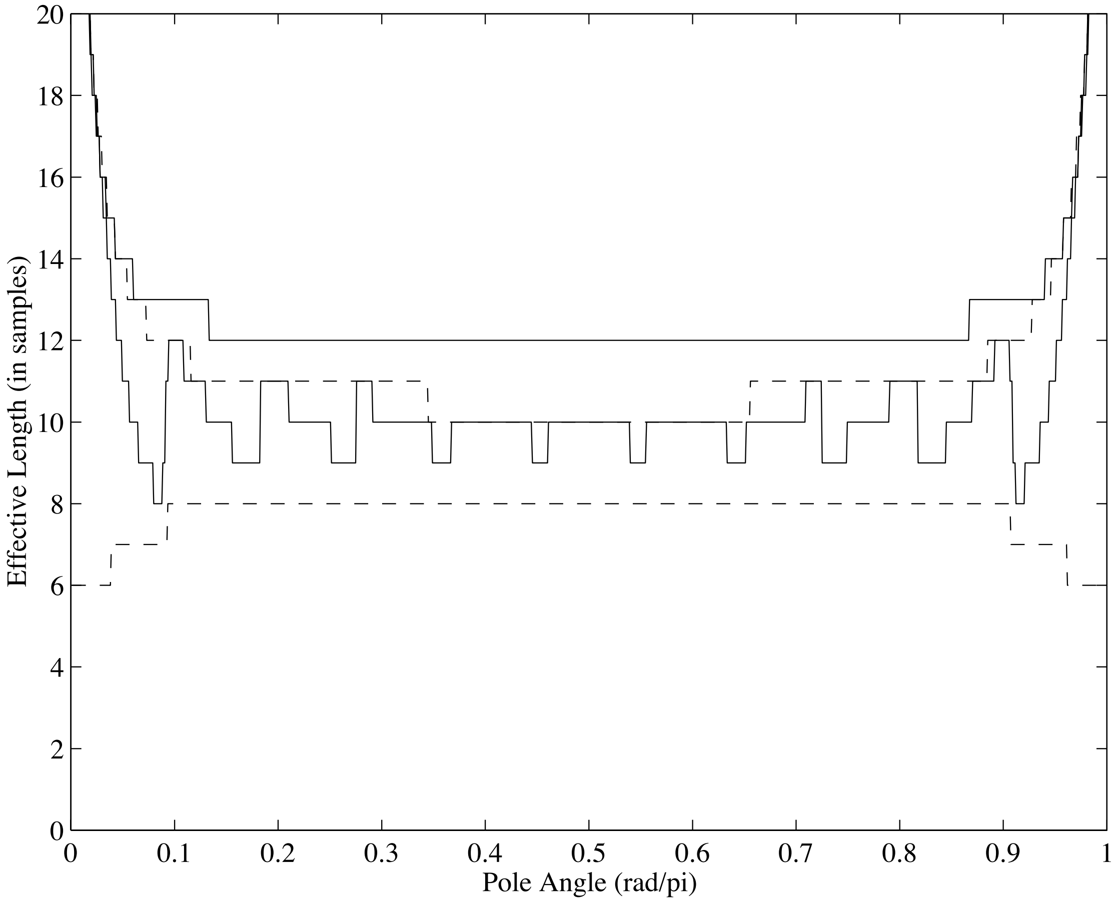

Fig. B.2. The effective length N_P (P = 90%) of the second-order all-pole filter as a function of parameter θ (r = 0.9). The solid line gives the actual value for N_P (obtained by numerical simulation) and the dashed lines present its upper and lower limit. The uppermost curve results from the simple approximation (B.13).

Closed-form formulas for the length of infinite length impulse responses of first and second-order recursive systems were derived. These measures are based on the idea of determining when P% of the total energy of the impulse response has arrived. Formulas for higher-order systems are more complicated but results can also be obtained by numerical simulation.

Simple Estimation of the Effective Length

The effective length of an infinite impulse response can also be estimated based on the time constant of the filter. This approach is used often in analog electronics. The time constant τ of the first-order all-pole filter can be solved in the following way

$$r^n = e^{-\frac{n}{\tau}} \tag{B.24}$$

where r is the radius of the pole, i.e., $r = |a_1|$. Now the time constant can be solved as

$$\tau = -\frac{1}{\ln(r)} \tag{B.25}$$

A simple approximation for the time constant can be derived by considering the Taylor series of the function ln(r), that is

$$\ln(r) = (r-1) - \frac{1}{2}(r-1)^2 + \frac{1}{3}(r-1)^3 - \ldots \tag{B.26}$$

When we truncate the first term of this series, we obtain the following approximation for the time constant of the one-pole filter†:

$$\tau = \frac{1}{1-r} \tag{B.27}$$

This is also an approximation for the effective length of the impulse response of a second-order all-pole filter. This is because the envelope of its impulse response decays as r^n [see Eq. (B.9)].

This simple approximation for the time constant can be used for estimating the length of the impulse response. For example, the impulse response has decayed about 60 dB in 7τ samples. More accurate estimates for the effective length of impulse responses can be computed using the formulas that were derived above utilizing the cumulated energy of the impulse response.

† J. O. Smith, personal communication, November 1995.