Appendix A

Bandlimited Squared Integral Error of FIR FD Filters

The least squared (LS) error function E_{LS} is the L_2 norm (integrated squared magnitude) of the error frequency response $E(e^{j\omega})$. In the following, we derive the bandlimited version of the error function E_{LS}, that is

$$E_{\mathrm{LS}} = \frac{1}{\pi}\int_0^{\alpha\pi} \left|E(e^{j\omega})\right|^2 d\omega = \frac{1}{\pi}\int_0^{\alpha\pi} \left|H(e^{j\omega}) - H_{\mathrm{id}}(e^{j\omega})\right|^2 d\omega \tag{A.1}$$

where $0 < \alpha \leq 1$. The error function can be rewritten using vectors and matrices, i.e.

$$\begin{aligned} E_{\mathrm{LS}} &= \frac{1}{\pi}\int_0^{\alpha\pi} \left[\mathbf{h}^T\mathbf{e} - H_{\mathrm{id}}(e^{j\omega})\right]\left[\mathbf{h}^T\mathbf{e} - H_{\mathrm{id}}(e^{j\omega})\right]^* d\omega \\ &= \frac{1}{\pi}\int_0^{\alpha\pi} \left[\mathbf{h}^T\mathbf{C}\mathbf{h} - 2\mathbf{h}^T \operatorname{Re}\left\{H_{\mathrm{id}}(e^{j\omega})\mathbf{e}^*\right\} + \left|H_{\mathrm{id}}(e^{j\omega})\right|^2\right] d\omega \end{aligned} \tag{A.2}$$

Here $\mathbf{h} = \begin{bmatrix} h(0) & h(1) & \cdots & h(N)\end{bmatrix}^T$ is the coefficient vector of the FIR filter and $\mathbf{e}$ is

$$\mathbf{e} = \begin{bmatrix} 1 & e^{-j\omega} & \cdots & e^{-jN\omega}\end{bmatrix}^T \tag{A.3}$$

and we have defined the matrix $\mathbf{C}$ as

$$\mathbf{C} = \operatorname{Re}\left\{\mathbf{e}\mathbf{e}^H\right\} = \begin{bmatrix} 1 & \cos(\omega) & \cdots & \cos(N\omega) \\ \cos(\omega) & 1 & & \cos[(N-1)\omega] \\ \vdots & & \ddots & \vdots \\ \cos(N\omega) & \cos[(N-1)\omega] & \cdots & 1 \end{bmatrix} \tag{A.4}$$

The squared error can be expressed as

$$E_{\mathrm{LS}} = \mathbf{h}^T\mathbf{P}\mathbf{h} - 2\mathbf{h}^T\mathbf{p}_1 + p_0 \tag{A.5a}$$

where we have used the following matrices and vectors

$$\mathbf{P} = \frac{1}{\pi}\int_0^{\alpha\pi} \mathbf{C}\, d\omega \tag{A.5b}$$

$$\mathbf{p}_1 = \frac{1}{\pi}\int_0^{\alpha\pi}\left[\mathrm{Re}\left\{H_{\mathrm{id}}(e^{j\omega})\right\}\mathbf{c} - \mathrm{Im}\left\{H_{\mathrm{id}}(e^{j\omega})\right\}\mathbf{s}\right]d\omega \tag{A.5c}$$

$$\mathbf{c} = \begin{bmatrix}1 & \cos(\omega) & \cdots & \cos(N\omega)\end{bmatrix}^T \tag{A.5d}$$

$$\mathbf{s} = \begin{bmatrix}0 & \sin(\omega) & \cdots & \sin(N\omega)\end{bmatrix}^T \tag{A.5e}$$

$$p_0 = \frac{1}{\pi}\int_0^{\alpha\pi}\left|H_{\mathrm{id}}(e^{j\omega})\right|^2 d\omega = \frac{1}{\pi}\int_0^{\alpha\pi} d\omega = \alpha \tag{A.5f}$$

The elements of matrix $\mathbf{P}$ can be elaborated as

$$\begin{aligned} P_{k,l} &= \frac{1}{\pi}\int_0^{\alpha\pi}\mathbf{C}d\omega = \frac{1}{\pi}\int_0^{\alpha\pi}\cos[(k-l)\omega]d\omega \\ &= \frac{\sin[(k-l)\alpha\pi]}{(k-l)\pi} = \alpha\mathrm{sinc}[(k-l)\alpha] \end{aligned} \tag{A.6}$$

for $k,l = 1,2,\ldots,N+1$. Also, the elements of vector $\mathbf{p}_1$ can be expressed as

$$\begin{aligned} p_{1,k} &= \frac{1}{\pi}\int_0^{\alpha\pi}[\cos(D\omega)\cos(k\omega) + \sin(D\omega)\sin(k\omega)]d\omega \\ &= \frac{1}{\pi}\int_0^{\alpha\pi}\cos[(D-k)\omega]d\omega = \frac{\sin[(D-k)\alpha\pi]}{[(D-k)\pi]} = \alpha\mathrm{sinc}[(D-k)\alpha] \end{aligned} \tag{A.7}$$

for $k = 1,2,\ldots,N+1$. Here we used the well-known trigonometric identities

$$\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$$

$$\sin(a)\sin(b) = \frac{1}{2}[\sin(a-b) - \sin(a+b)]$$

The second term of (A.5) can be further elaborated as

$$2\mathbf{h}^T\mathbf{p}_1 = 2\alpha\sum_{n=0}^{N}h(n)\mathrm{sinc}[(n-D)\alpha] \tag{A.8}$$

Finally, the bandlimited squared frequency response error can be expressed as a function of filter coefficients:

$$E_{\mathrm{LS}} = \mathbf{h}^T\mathbf{P}\mathbf{h} - 2\alpha\sum_{n=0}^{N}h(n)\mathrm{sinc}[(n-D)\alpha] + \alpha \tag{A.9}$$

where the elements of matrix $\mathbf{P}$ are given by (A.6).