Appendix A

Bandlimited Squared Integral Error of FIR FD Filters

The least squared (LS) error function $E_{LS}$ is the $L_2$ norm (integrated squared magnitude) of the error frequency response $E(e^{j\omega})$. In the following, we derive the bandlimited version of the error function $E_{LS}$, that is

$$E_{LS} = \frac{1}{\pi} \int_0^{\alpha \pi} \left| E(e^{j\omega}) \right|^2 d\omega = \frac{1}{\pi} \int_0^{\alpha \pi} \left| H(e^{j\omega}) - H_{id}(e^{j\omega}) \right|^2 d\omega$$  \hspace{1cm} (A.1)

where $0 < \alpha \leq 1$. The error function can be rewritten using vectors and matrices, i.e.

$$E_{LS} = \frac{1}{\pi} \int_0^{\alpha \pi} \left[ h^T e - H_{id}(e^{j\omega}) \right] \left[ h^T e - H_{id}(e^{j\omega}) \right]^* d\omega$$

$$= \frac{1}{\pi} \int_0^{\alpha \pi} \left[ h^T C h - 2 h^T \text{Re} \left\{ H_{id}(e^{j\omega}) e^s \right\} + \left| H_{id}(e^{j\omega}) \right|^2 \right] d\omega$$  \hspace{1cm} (A.2)

Here $h = [h(0) \ h(1) \cdots h(N)]^T$ is the coefficient vector of the FIR filter and $e$ is

$$e = [1 \ e^{−j\omega} \cdots e^{−jN\omega}]^T$$  \hspace{1cm} (A.3)

and we have defined the matrix $C$ as

$$C = \text{Re} \{ ee^T \} = \begin{bmatrix}
1 & \cos(\omega) & \cdots & \cos(N\omega) \\
\cos(\omega) & 1 & \cdots & \cos([N-1]\omega) \\
\vdots & \ddots & \ddots & \vdots \\
\cos(N\omega) & \cos([N-1]\omega) & \cdots & 1
\end{bmatrix}$$  \hspace{1cm} (A.4)

The squared error can be expressed as

$$E_{LS} = h^T Ph - 2h^T p_1 + p_0$$  \hspace{1cm} (A.5a)

where we have used the following matrices and vectors

$$P = \frac{1}{\pi} \int_0^{\alpha \pi} C d\omega$$  \hspace{1cm} (A.5b)
\[
\mathbf{p}_1 = \frac{1}{\pi} \int_0^{\alpha \pi} \left[ \text{Re}\{H_{id}(e^{j\omega})\}e - \text{Im}\{H_{id}(e^{j\omega})\}s \right] d\omega
\] (A.5c)

\[
c = [1 \quad \cos(\omega) \quad \cdots \quad \cos(N\omega)]^T
\] (A.5d)

\[
s = [0 \quad \sin(\omega) \quad \cdots \quad \sin(N\omega)]^T
\] (A.5e)

\[
p_0 = \frac{1}{\pi} \int_0^{\alpha \pi} \left| H_{id}(e^{j\omega}) \right|^2 d\omega = \frac{1}{\pi} \int d\omega = \alpha
\] (A.5f)

The elements of matrix \( \mathbf{P} \) can be elaborated as

\[
P_{k,l} = \frac{1}{\pi} \int_0^{\alpha \pi} C d\omega = \frac{1}{\pi} \int_0^{\alpha \pi} \cos((k-l)\omega) d\omega
\]

\[
= \frac{\sin[(k-l)\alpha\pi]}{(k-l)\pi} = \alpha \text{sinc}[(k-l)\alpha]
\] (A.6)

for \( k,l = 1,2,\ldots,N+1 \). Also, the elements of vector \( \mathbf{p}_1 \) can be expressed as

\[
p_{l,k} = \frac{1}{\pi} \int_0^{\alpha \pi} \left[ \cos(D\omega)\cos(k\omega) + \sin(D\omega)\sin(k\omega) \right] d\omega
\]

\[
= \frac{1}{\pi} \int_0^{\alpha \pi} \cos((D-k)\omega) d\omega = \frac{\sin[(D-k)\alpha\pi]}{(D-k)\pi} = \alpha \text{sinc}[(D-k)\alpha]
\] (A.7)

for \( k,l = 1,2,\ldots,N+1 \). Here we used the well-known trigonometric identities

\[
\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]
\]

\[
\sin(a)\sin(b) = \frac{1}{2}[\sin(a-b) - \sin(a+b)]
\]

The second term of (A.5) can be further elaborated as

\[
2\mathbf{h}^T \mathbf{p}_1 = 2\alpha \sum_{n=0}^{N} h(n)\text{sinc}[(n-D)\alpha]
\] (A.8)

Finally, the bandlimited squared frequency response error can be expressed as a function of filter coefficients:

\[
E_{LS} = \mathbf{h}^T \mathbf{P} \mathbf{h} - 2\alpha \sum_{n=0}^{N} h(n)\text{sinc}[(n-D)\alpha] + \alpha
\] (A.9)

where the elements of matrix \( \mathbf{P} \) are given by (A.6).