3.4 Design Methods for Fractional Delay Allpass Filters

Above we have studied the design of FIR filters for fractional delay approximation. Now we show how recursive or IIR filters can be used for the same problem. Since the magnitude response of an ideal FD element is perfectly flat, we consider only the so-called *allpass filters*†. Their magnitude response is always flat irrespective of the filter coefficients. Allpass filters are typically used for phase equalization and other signal processing tasks where the phase characteristics are of greatest concern. Also the FD approximation is essentially a phase approximation problem and thus the allpass filter is particularly well suited to this task.

3.4.1 Properties of the Discrete-Time Allpass Filter

A discrete-time allpass filter has the transfer function

$$A(z) = \frac{z^{-N}D(z^{-1})}{D(z)} = \frac{a_N + a_{N-1}z^{-1} + \ldots + a_1 z^{-(N-1)} + z^{-N}}{1 + a_1 z^{-1} + \ldots + a_{N-1}z^{-(N-1)} + a_N z^{-N}} \tag{3.117}$$

where N is the order of the filter, $D(z)$ is the denominator polynomial, and the filter coefficients a_k (k = 1, 2, ..., N) are real. The block diagram of the allpass filter is depicted in Fig. 3.19.

The numerator of the allpass transfer function is a mirrored version of the denominator polynomial. The poles (i.e., roots of the denominator) of a stable allpass filter are located inside the unit circle in the complex plane and as a result its zeros (i.e., roots of the numerator) are located outside the unit circle so that their angle is the same but the radius is the inverse of the corresponding pole. For this reason the magnitude response of an allpass filter is flat. It is expressed as

$$\left|A(e^{j\omega})\right| = \left|\frac{e^{-j\omega N}D(e^{-j\omega})}{D(e^{j\omega})}\right| \equiv 1 \tag{3.118}$$

Obviously, the name ‘allpass filter’ comes from the above property that this filter passes signal components of all frequencies without attenuating or boosting them.

The frequency response of the allpass filter can be expressed in the form

$$A(e^{j\omega}) = e^{j\Theta_{\mathrm{A}}(\omega)} \tag{3.119}$$

This form stresses the fact that the main feature of an allpass filter is its *phase response* $\Theta_A(\omega)$. It can be written as

$$\Theta_A(\omega) = \arg\left\{A(e^{j\omega})\right\} = -N\omega + 2\Theta_D(\omega) \tag{3.120}$$

where $\Theta_D(\omega)$ is the phase response of $1\big/D(e^{j\omega})$, that is

† The term ‘allpass filter’ is sometimes used to denote any discrete-time system (FIR or IIR) whose magnitude response is almost flat at all frequencies ($|H(e^{j\omega})| \approx 1$). Throughout this text this term is used only for recursive discrete-time filters which have the exact allpass property, that is, $|H(e^{j\omega})| \equiv 1$.

$$\Theta_D(\omega) = \arg\left\{\frac{1}{D(e^{j\omega})}\right\} = \arctan\left\{\frac{\sum_{k=0}^{N} a_k \sin(k\omega)}{\sum_{k=0}^{N} a_k \cos(k\omega)}\right\} = \arctan\left\{\frac{\mathbf{a}^T\mathbf{s}}{\mathbf{a}^T\mathbf{c}}\right\} \tag{3.121a}$$

with $a_0 = 1$ and the vectors **a**, **s**, and **c** are defined by

$$\mathbf{a} = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_N \end{bmatrix}^T \tag{3.121b}$$

$$\mathbf{s} = \begin{bmatrix} 0 & \sin\omega & \sin(2\omega) & \cdots & \sin(N\omega) \end{bmatrix}^T \tag{3.121c}$$

$$\mathbf{c} = \begin{bmatrix} 1 & \cos\omega & \cos(2\omega) & \cdots & \cos(N\omega) \end{bmatrix}^T \tag{3.121d}$$

The *phase delay* $\tau_{\mathrm{p},A}(\omega)$ of an allpass filter can be expressed with the help of Eq. (3.120) as

$$\tau_{\mathrm{p},A}(\omega) = -\frac{\Theta_A(\omega)}{\omega} = N - 2\tau_{\mathrm{p},D}(\omega) \tag{3.122}$$

where $\tau_{\mathrm{p},D}(\omega)$ is the phase delay of $1/D(e^{j\omega})$. The corresponding *group delay* is given by

$$\tau_{\mathrm{g},A}(\omega) = -\frac{d\Theta_A(\omega)}{d\omega} = N - 2\tau_{\mathrm{g},D}(\omega) \tag{3.123}$$

where $\tau_{\mathrm{g},D}(\omega)$ is the group delay of $1/D(e^{j\omega})$.

3.4.2 Design of Fractional Delay Allpass Filters

There is a noteworthy difference between the design of FIR and allpass filters: the coefficients of an FIR filter are easily obtained by the inverse discrete-time Fourier transform of the frequency-domain specifications, since the coefficients of an FIR filter are equal to the samples of its impulse response. However, the relationship between the transfer function coefficients and the impulse response of an allpass filter is not this simple. Hence, most of the design techniques for allpass filters are iterative. Many of the design methods for FD allpass filters are counterparts of FIR design methods discussed in Section 3.2.

The desired or ideal phase response in the fractional delay approximation problem is

$$\Theta_{\mathrm{id}}(\omega) = -D\omega \tag{3.124}$$

as may be seen from the ideal frequency response given by Eq. (3.8). The phase error of an allpass filter can be defined as the deviation from the desired phase function $\Theta_{\mathrm{id}}(\omega)$ as

$$\Delta\Theta(\omega) = \Theta_{\mathrm{id}}(\omega) - \Theta_A(\omega) = 2\arctan\left\{\frac{\mathbf{a}^T\mathbf{s}_\beta}{\mathbf{a}^T\mathbf{c}_\beta}\right\} \tag{3.125}$$

where **a** is the coefficient vector given by Eq. (3.121b), $\mathbf{s}_\beta$ and $\mathbf{c}_\beta$ are defined as

$$\mathbf{s}_\beta = \left[\sin\{\beta(\omega)\} \quad \sin\{\beta(\omega)-\omega\} \quad \cdots \quad \sin\{\beta(\omega)-N\omega\}\right]^T \tag{3.126a}$$

$$\mathbf{c}_\beta = \left[\cos\{\beta(\omega)\} \quad \cos\{\beta(\omega)-\omega\} \quad \cdots \quad \cos\{\beta(\omega)-N\omega\}\right]^T \tag{3.126b}$$

and $\beta(\omega)$ as

$$\beta(\omega) = \frac{1}{2}\left[\Theta_{\mathrm{id}}(\omega) + N\omega\right] \tag{3.126c}$$

When approximating a fractional delay $D = N + d$, or the ideal phase function $\Theta_{\mathrm{id}}(\omega) = -D\omega = -(N+d)\omega$, the last form reduces to

$$\beta(\omega) = -\frac{\omega d}{2} \tag{3.127}$$

The weighted least-squares *phase* error that is to be minimized is defined as

$$E_{\mathrm{LS}} = \frac{1}{\pi}\int_0^{\pi} W(\omega)\left|\Delta\Theta(\omega)\right|^2 d\omega \tag{3.128}$$

where $W(\omega)$ is a nonnegative weighting function. Lang and Laakso (1994) have introduced an iterative algorithm for the design of the coefficient vector **a**. The algorithm typically converges to the desired solution, but this cannot be guaranteed.

The LS *phase delay* error can be defined as

$$\begin{aligned} E_{\mathrm{LS}} &= \frac{1}{\pi}\int_0^{\pi} W(\omega)\left|\Delta\tau_{\mathrm{p}}(\omega)\right|^2 d\omega = \frac{1}{\pi}\int_0^{\pi} W(\omega)\left|\frac{\Delta\Theta(\omega)}{\omega}\right|^2 d\omega \\ &= \frac{1}{\pi}\int_0^{\pi} \frac{W(\omega)}{\omega^2}\left|\Delta\Theta(\omega)\right|^2 d\omega \end{aligned} \tag{3.129}$$

In other words, the phase delay error solution is obtained by introducing an additional weighting function $W@\omega) = 1/\omega^2$ to the phase error, Eq. (3.128). Allpass filters may be designed using the iterative algorithm modified from that mentioned above.

There exists a large variety of algorithms for equiripple or minimax allpass filter design in terms of phase, group delay, or phase delay (see Laakso *et al.*, 1994 for references). New iterative techniques for equiripple phase error and phase delay error design are described in Laakso *et al.* (1994).

It appears that, just as with the FD FIR filters, a well suited digital allpass filter for FD approximation in audio signal processing is obtained by using the maximally flat design at $\omega = 0$. This technique deserves more attention than the others.

3.4.3 Maximally Flat Group Delay Design of FD Allpass Filters

The maximally flat group delay design is the only known FD allpass filter design technique that has a closed-form solution (Laakso *et al.*, 1992, 1994). Here we discuss this solution.

Thiran (1971) proposed an analytic solution for the coefficients of an all-pole lowpass filter with a maximally flat group delay response at the zero frequency

$$a_k = (-1)^k \binom{N}{k} \prod_{n=0}^{N} \frac{2d+n}{2d+k+n} \quad \text{for } k = 0,1,2,\ldots,N \tag{3.130}$$

with the binomial coefficient

$$\binom{N}{k} = \frac{N!}{k!(N-k)!} \tag{3.131}$$

It has been shown by Thiran (1971) that when $d > 0$ the denominator polynomial obtained by the above method will have its roots within the unit circle in the complex plane, i.e., the filter will be *stable*. A drawback of Thiran's design technique is that the magnitude response of the all-pole lowpass filter cannot be controlled. In the allpass design, however, this kind of problem does not exist. Thus it seems that the result of Thiran is better suited to the design of allpass than all-pole filters.

As can be seen from Eq. (3.123) the group delay of an allpass filter is twice that of its all-pole counterpart. Thus Thiran's solution may easily be applied to the design of all-pass filters by making the substitution $d' = d/2$. The solution for the coefficients of a *maximally flat* (MF) *allpass filter* is thus (Laakso *et al.*, 1992)

$$a_k = (-1)^k \binom{N}{k} \prod_{n=0}^{N} \frac{d+n}{d+k+n} \quad \text{for } k = 0,1,2,\ldots,N \tag{3.132}$$

Note that $a_0 = 1$ always and thus the coefficient vector **a** need not be scaled. This closed-form solution to the allpass filter approximation is called the *Thiran allpass filter*.

The allpass filters designed using Eq. (3.132) approximate a group delay of $N + d$ samples. This kind of parametrization is rather impractical. For this reason we substitute $d = D - N$ into Eq. (3.132). As a result we obtain the following formula:

$$a_k = (-1)^k \binom{N}{k} \prod_{n=0}^{N} \frac{D-N+n}{D-N+k+n} \quad \text{for } k = 0,1,2,\ldots,N \tag{3.133}$$

Now the delay parameter D refers to the actual delay rather than the offset from N samples. In Table 3.1 we give the coefficients for the Thiran allpass filters with $N = 1$, 2, and 3.

Table 3.1 Coefficients of the Thiran allpass filters of order $N = 1$, 2, and 3.

	a_1	a_2	a_3
$N = 1$	$-\frac{D-1}{D+1}$		
$N = 2$	$-2\frac{D-2}{D+1}$	$\frac{(D-1)(D-2)}{(D+1)(D+2)}$	
$N = 3$	$-3\frac{D-3}{D+1}$	$3\frac{(D-2)(D-3)}{(D+1)(D+2)}$	$-\frac{(D-1)(D-2)(D-3)}{(D+1)(D+2)(D+3)}$

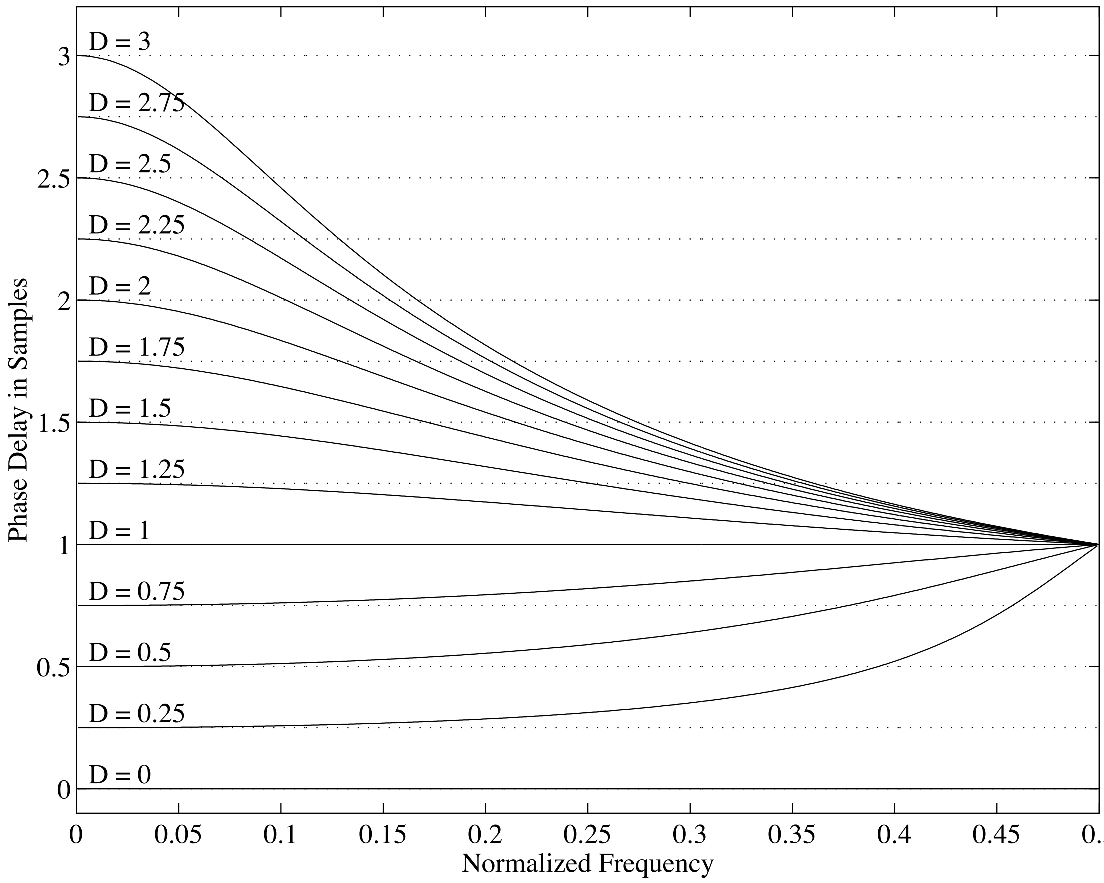

Fig. 3.20 The phase delay of a first-order ($N = 1$) Thiran allpass filter. The dashed lines indicate the ideal phase delay which is constant.

Thiran's proof of stability implies that this allpass filter will be stable when $D > N$. We have experimentally observed that the Thiran allpass filter is also stable when the delay is $N - 1 < D < N$. At $D = N - 1$, one of the poles (as well as one zero) of the Thiran allpass filter will be on the unit circle making the filter asymptotically unstable. When $D < N - 1$, one or more poles will be outside the unit circle and the filter will be unstable.

In Figs. 3.20, 3.21, and 3.22, the phase delay responses of first, second, and third-order MF allpass interpolators are presented for several values of the delay parameter D starting at $D = N - 1$. The ideal phase delay (constant at all frequencies) is illustrated with a dotted line in each figure. It is seen that the Thiran allpass filter is most accurate at low frequencies. This is an expected result since Thiran designed the denominator polynomial so that the group delay and its N derivatives evaluated at the zero frequency match the desired result. As the filter order is increased, the phase delay curves remain nearly constant at higher frequencies.

The closed-form design of the first-order allpass filter for varying the length of a delay line has been discussed in papers by Jaffe and Smith (1983) and Smith and Friedlander (1985). They obtained the solution by first expressing the phase response of the first-order allpass filter by its Taylor series with the expansion point $z = 0$ and then truncating this series after the first term. The expression for the filter coefficient a_1 of the first-order ($N = 1$) allpass filter is the same as that given in Table 3.1.

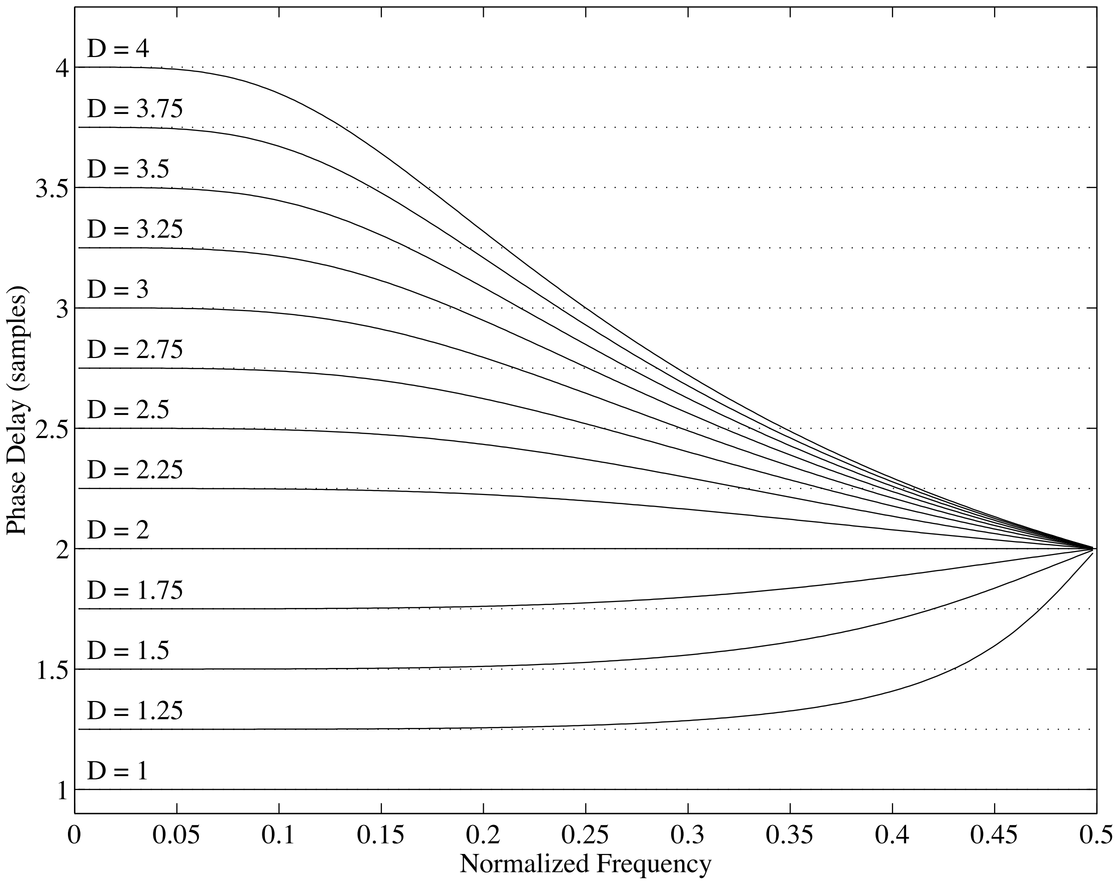

Fig. 3.21 The phase delay of a second-order ($N = 2$) Thiran allpass filter.

3.4.4 Choice of Optimal Range for D in the Thiran Interpolator

In the case of FIR FD filters we noticed that it is a good choice to use values $(N - 1)/2 \leq D < (N + 1)/2$, since then the mean squared error is the smallest in all cases of d. In the case of recursive filters this rule does not apply. In the following we analyze the optimal range for the delay parameter in the case of the Thiran allpass filter.

Let us examine the mean squared error function of the Thiran allpass filter given by

$$\begin{aligned} E_{\mathrm{S}} &= \frac{1}{\pi}\int_0^{\pi} \left|E(e^{j\omega})\right|^2 d\omega = \frac{1}{\pi}\int_0^{\pi} \left|A(e^{j\omega}) - H_{\mathrm{id}}(e^{j\omega})\right|^2 d\omega \\ &= \frac{1}{\pi}\int_0^{\pi} \left|e^{j\Theta_A(\omega)} - e^{-j\omega D}\right|^2 d\omega = \frac{1}{\pi}\int_0^{\pi} \left[e^{j\Theta_A(\omega)} - e^{-j\omega D}\right]\left[e^{-j\Theta(\omega)} - e^{j\omega D}\right] d\omega \\ &= \frac{2}{\pi}\left\{1 - \int_0^{\pi} \cos\left[\Theta_A(\omega) + \omega D\right] d\omega\right\} \end{aligned} \tag{3.134}$$

Integration of the last form is difficult and we thus use numerical methods for investigating the approximation error. We use a numerical integration technique to evaluate the error for Thiran allpass filters of different order. Figure 3.23 gives the error E_{S} as a

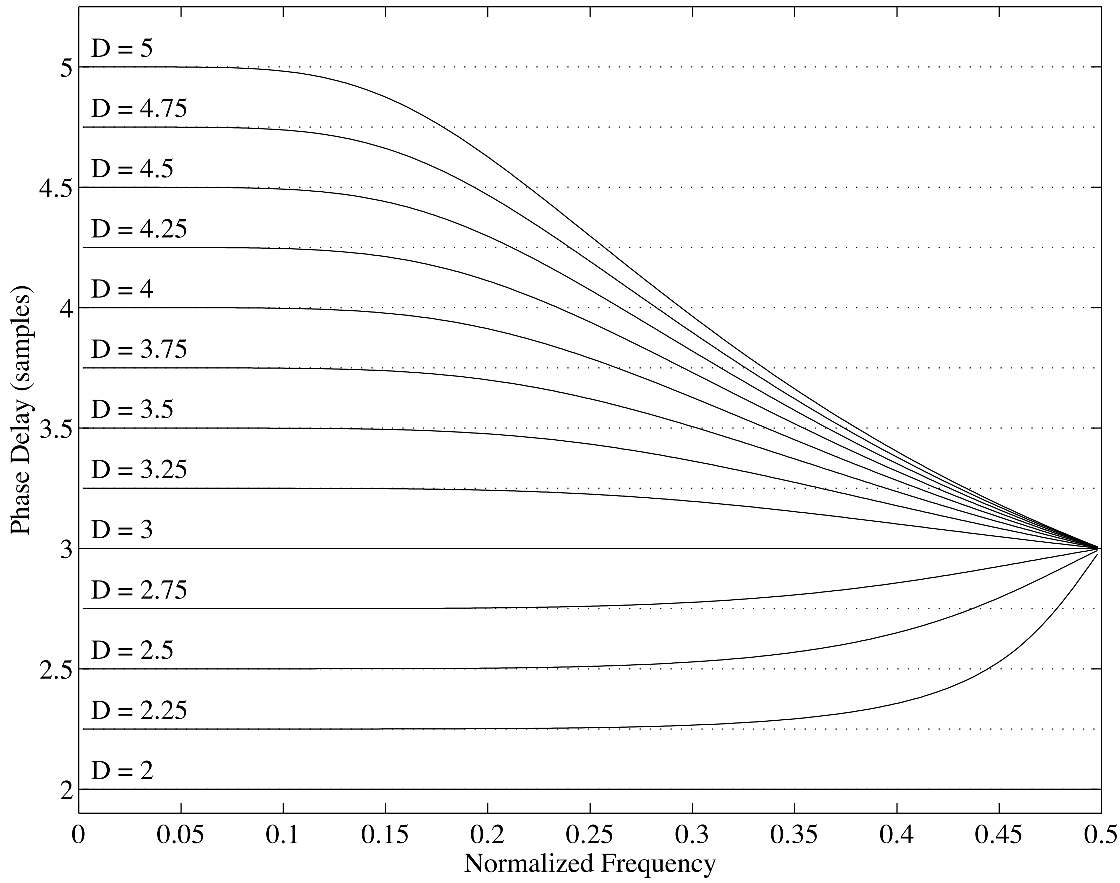

Fig. 3.22 The phase delay of a third-order ($N = 3$) Thiran allpass filter.

function of the delay parameter D for Thiran allpass filters of orders 1 to 6.

It is seen that the error is zero for integral delay values $N - D = -1$ or 0. This implies that the Thiran allpass filter yields the original signal samples in these cases. As discussed before, the case $N - D = -1$ yields an asymptotically unstable filter. It is suggested not to use this kind of filter in practical work. The Thiran allpass filter can thus only be used for delay approximations such that $D > N - 1$.

In the case $D = N$ the Thiran filter reduces to a delay line because all the coefficients of the numerator polynomial are zero. This is easily shown by noticing that each coefficient computed according to (3.133) has a factor $D - N$ in its numerator.

A fractional delay filter can usually be used for approximating a delay D for which it holds

$$D_0 \le D < D_0 + 1 \tag{3.135}$$

To solve for the optimal choice for D_0 we may evaluate the average error E_{ave} written as

$$E_{\mathrm{ave}}(D_0) = \int_{D_0}^{D_0+1} E_{\mathrm{S}}(D)dD \tag{3.136}$$

We can evaluate this integral numerically. Figure 3.24 presents the average error as a function of D_0 for some low-order Thiran allpass filters. The minimum of each curve is indicated by a circle in Fig. 3.24. These points correspond to the optimal D_0 for each

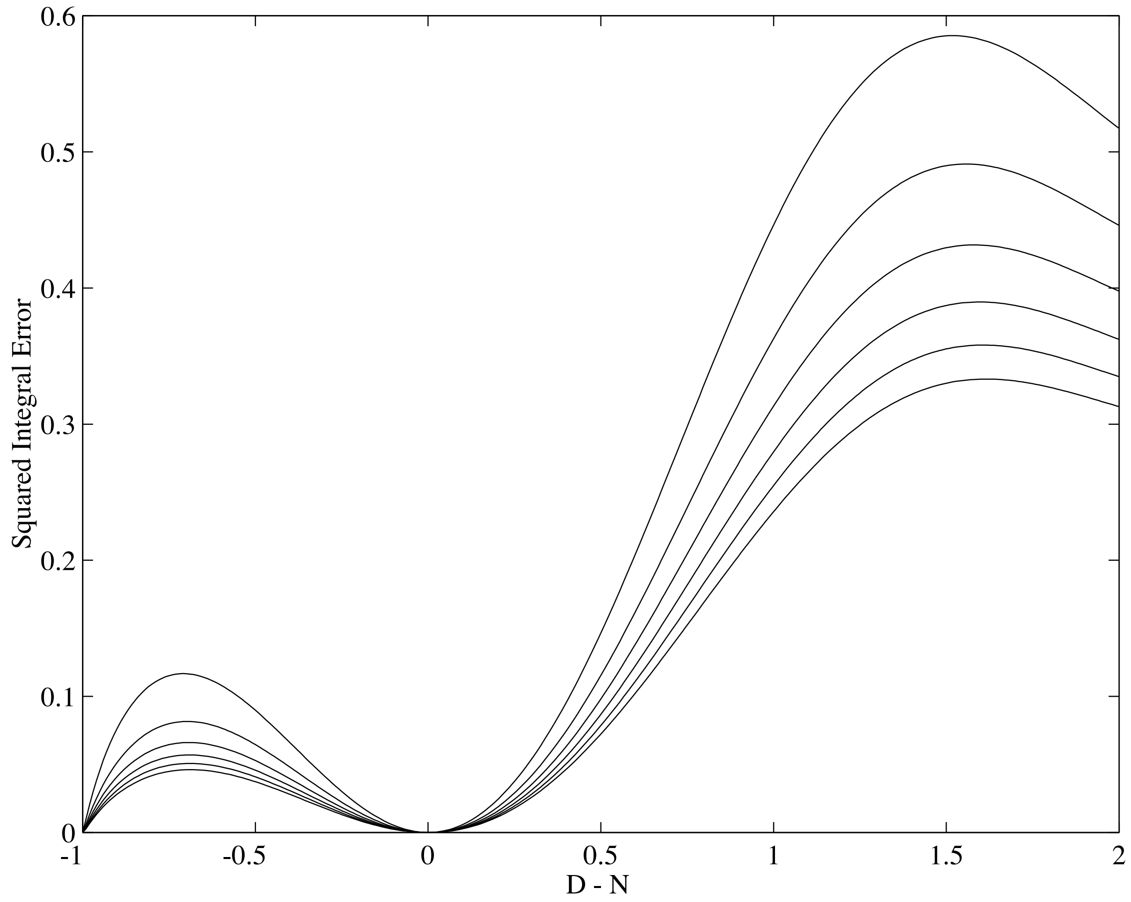

Fig. 3.23 Squared integral error of Thiran allpass filters of orders $N = 1$ to 6 (top to bottom).

filter. The values of D_0 and the corresponding average error E_{ave} are presented in Table 3.2.

Some general observations can be made from Fig. 3.24. Apparently, the approximation error of the Thiran allpass filter is quite small for delay values $N - 1 < D < N$ reaching a minimum at the value given in Table 3.2. For much larger values of D, the approximation error increases considerably. It is seen in Fig. 3.24 that the average error does not increase very rapidly in the neighborhood of the minimum. Thus, if the best possible accuracy is not essential, one may use a simple rule of thumb to choose the

Table 3.2 The optimal values for D_0 and minimum average errors for low-order Thiran allpass filters. $E_{\mathrm{ave}}(N - 0.5)$ is the average error when $D_0 = N - 0.5$.

N	$D_{0,\,\mathrm{opt}}$	$E_{\mathrm{ave}}(D_{0,\,\mathrm{opt}})$	$E_{\mathrm{ave}}(N - 0.5)$
1	0.418	0.040	0.043
2	1.403	0.030	0.033
3	2.396	0.025	0.027
4	3.392	0.022	0.024
5	4.390	0.019	0.022
6	5.389	0.018	0.020

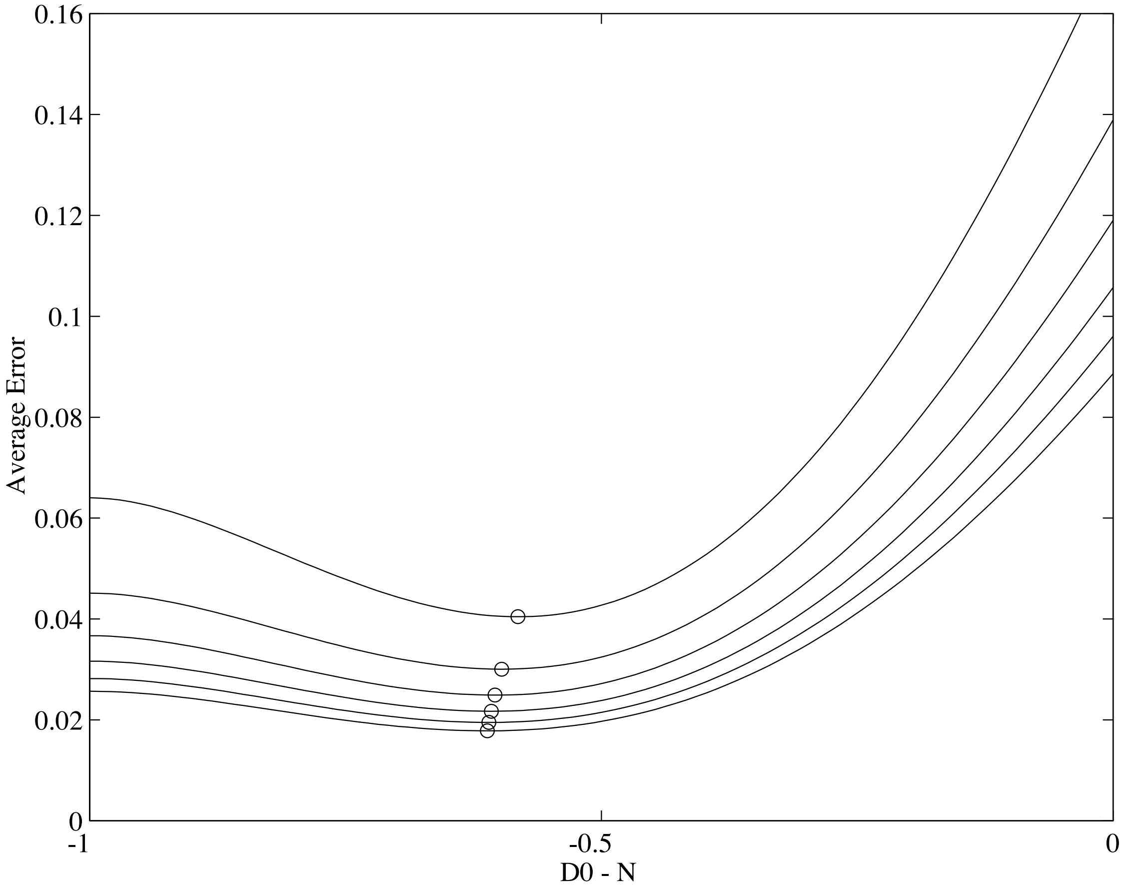

Fig. 3.24 The average error of low-order Thiran allpass filters as a function of D_0. The filter orders from top to bottom are N = 1, 2, 3, 4, 5, and 6. The minimum of each error function is indicated by a circle.

range for D: for the Thiran allpass filter, a close-to-minimum error is obtained when $D_0 = N - 0.5$, which means that D is

$$N - 0.5 \leq D < N + 0.5 \qquad (3.137)$$

A similar statement as in the case of FIR FD filters can be made: the total delay of the Thiran allpass filters increases linearly with order N.

3.4.5 Discussion

This section has dealt with allpass filter approximation of fractional delay. Digital allpass filters are a good choice for this task since their magnitude response is exactly flat and the design can concentrate on the phase properties. The Thiran interpolator was discussed in detail since it is easy to design with closed-form formulas and it is very accurate at low frequencies.

The design of general IIR filters (with different numerator and denominator polynomials) for FD approximation has not been much discussed in the DSP literature. One example is the paper by Tarczynski and Cain (1994) where reduced-bandwidth IIR filters are optimized iteratively. The design of optimal IIR filters is a difficult problem but new techniques are certainly worth trying since generally speaking recursive filters are more powerful than FIR filters. This is a fascinating topic for future research.