MIXED PHYSICAL MODELING: DWG + FDTD + WDF

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ABSTRACT

Physical modeling by computer, especially for sound synthesis of musical instruments, is based on discrete time and space numerical simulation of the targeted physical systems. Digital waveguides have been found a particularly attractive technique due to their efficiency and inherent capability of modeling systems with propagation delays. Other techniques used are for example finite difference schemes, wave digital filters, modal synthesis techniques, etc. The relation between these approaches has been investigated to some degree, but systematic studies on how to build mixed models including subsystems of different types remain to be carried out. The aim of this paper is to explore some of the relationships between particular modeling paradigms and to show how to construct flexible yet efficient models for sound synthesis.

1. INTRODUCTION

This paper extends the work, initiated in papers [1, 2, 3, 4], towards generalized and mixed modeling, using different physical modeling paradigms. The similarity and differences between digital waveguides [5, 6] and finite difference time domain schemes [7, 8, 9] drew our attention to the possibility of making hybrids of these approaches. Additionally, adding lumped elements by wave digital filters [10, 6] extends this framework of physical modeling. Finally it turns out that this makes a time-domain simulator for circuits and networks composed of parametrically controlled discrete time and space elements. It will be shown in this paper how a DSP-based formulation of these computational blocks can lead to a systematic and flexible yet computationally efficient formalism for mixed physical modeling.

2. DISCRETE-TIME PHYSICAL MODELING

In time-domain discrete-time modeling of physical systems the task is to convert the underlying (partial) differential equations into approximating difference equations then to be solved. Formulation of the solution as DSP algorithms makes them computable by efficient software tools, even as real-time simulation. In this paper we next introduce DSP formulations of the two main approaches of interest, the digital waveguides (DWG) and the finite difference time-domain schemes (FDTD). Then we add wave digital filters (WDF) to this modeling framework.

In a one-dimensional lossless medium the wave equation is written

$$y_{tt} = c^2 y_{xx} \tag{1}$$

where y is a wave variable, subscript tt refers to second partial derivative in time t, xx to second partial derivative in place variable x, and c is speed of wavefront in the medium of interest.

2.1. Wave-based modeling

The 1-D traveling wave formulation is based on the d'Alembert solution of propagation of two opposite direction waves, i.e.,

$$y(t,x) = \overrightarrow{y}(t - x/c) + \overleftarrow{y}(t + x/c) \tag{2}$$

where the arrows denote the right-going and the left-going components of the total waveform. Assuming that the signals are bandlimited to half of sampling rate, the traveling waves can be sampled without losing any information by selecting T as the sample interval and X the position interval between samples so that $T = X/c$. Sampling is applied in a discrete time-space grid in which n and m are related to time and position, respectively. The discretized version of Eq. (2) [5] becomes:

$$y(n,m) = \overrightarrow{y}(n - m) + \overleftarrow{y}(n + m) \tag{3}$$

It follows that the wave propagation can be computed by updating state variables in two delay lines by

$$\overrightarrow{y}_{k,n+1} = \overrightarrow{y}_{k-1,n} \quad \text{and} \quad \overleftarrow{y}_{k,n+1} = \overleftarrow{y}_{k+1,n} \tag{4}$$

i.e., by simply shifting the samples to the right and left, respectively. This kind of discrete-time modeling is called Digital Waveguide (DWG) modeling [5]. Since the physical wave variables are split explicitly into directional wave components, we will call such models *W-models*.

The next step is to take into account the global physical constraints of continuity by Kirchhoff type of rules. This means to formulate the scattering junctions of interconnected ports, with given impedances and wave variables at related ports. For a scattering junction, where the physical variables are sound wave pressure P and volume velocity U, and when a parallel admittance model [1] of N ports is utilized, the Kirchhoff constrains become

$$P_1 = P_2 = \ldots = P_N = P_J \tag{5}$$

$$U_1 + U_2 + \ldots + U_N + U_{\text{ext}} = 0 \tag{6}$$

where P_J is the common pressure of coupled branches and U_{ext} is an external volume velocity to the junction. When port pressures are represented by incoming wave components P_i^+, outgoing wave components by P_i^-, admittances attached to each port by Y_i, and

$$P_i = P_i^+ + P_i^- \quad \text{and} \quad U_i^+ = Y_i P_i^+ \tag{7}$$

the junction pressure P_J can be obtained as:

$$P_J = \frac{1}{Y_{\text{tot}}}(U_{\text{ext}} + 2\sum_{i=0}^{N-1} Y_i P_i^+) \tag{8}$$

[1] Models can be formulated as well for impedances instead of admittances and for series connection, and different physical variable pairs can be used.

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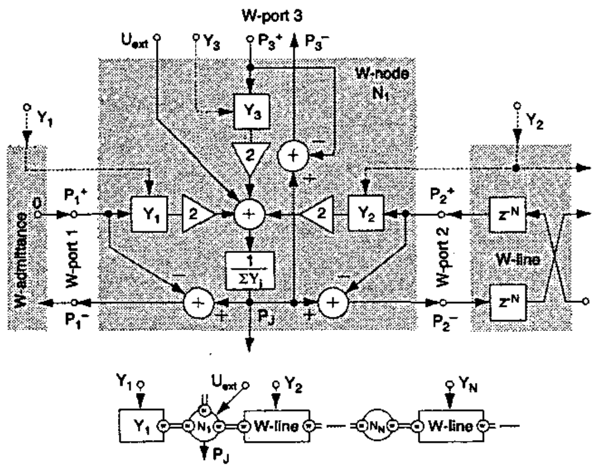

Figure 1: Top: A 3-port scattering junction (W-node N_1). Incoming pressures are P_i^+ and outgoing ones P_i^+. W-port 1 is connected to termination W-admittance Y_1 and port 2 to a two-directional delay line (W-line). Admittance controls are marked by dashed lines. Bottom: Block diagram with abstracted blocks and how they can be connected to form a 1-D DWG waveguide.

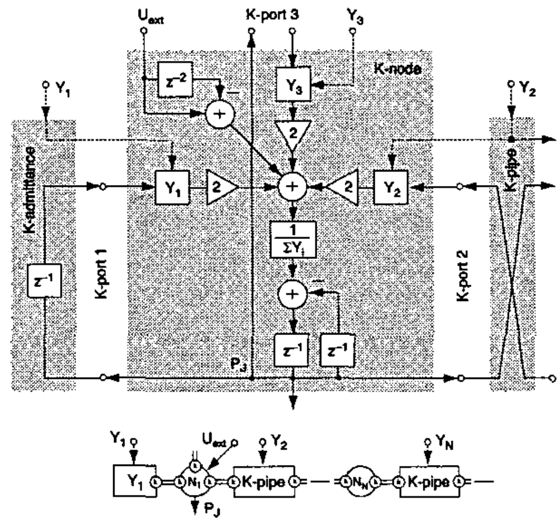

Figure 2: Top: Digital filter structure for finite difference approximation of a two-port scattering node with port admittances Y_1 and Y_2. Only total pressure P_J (K-variable) is explicitly available. Bottom: Block diagram with abstracted blocks and how they can be connected to form a 1-D FDTD waveguide.

where $Y_{tot} = \sum_{i=0}^{N-1} Y_i$ is the sum of all admittances to the junction. Outgoing pressure waves, obtained from Eq. (7), are then $P_i^- = P_J - P_i^+$. The resulting junction, a *W-node*, is depicted as a DSP structure in the N_1 node of Fig. 1 (top). When admittances Y_i are frequency-dependent, this diagram can be interpreted as a filter structure where the incoming pressures are filtered by the corresponding wave admittances Y_i times two, and their sum is filtered further by $1/Y_{\text{tot}}$ to get the junction pressure P_J.

Two special cases can be noticed on the basis of Eq. (8). First, a (passive) loading admittance is the case with Y_i where no incoming pressure wave component P_i^+ is associated. This needs no computation except including Y_i in Y_{tot} because $P_i^+ = 0$, see the left-hand termination, a *W-admittance*, in Fig. 1. Another issue is the external velocity U_{ext} effective to the junction. This is connected directly to the summation at the junction node.

The W-node in Fig. 1 is coupled through *W-ports* to the neighboring elements (port 3 is uncoupled). The right-hand side block is a two-directional delay line, a *W-line*, of admittance Y_2. The bottom part of the figure depicts a block diagram abstraction of the DSP structure. It also characterizes how waveguides are built as structures of W-line elements connected by W-node junctions.

Notice that the admittances in Fig. 1 may be real-valued or frequency-dependent so that Y_i and the impedance $1/\sum Y_i$ can be realized as FIR or IIR filters, or just as real coefficients if all attached admittances are real. In the latter case, if we skip the external velocity U_{ext} of Eq. (8), we may write the equation using scattering parameters α_i as $P_J = \sum_{i=0}^{N-1} \alpha_i P_i^+$, where $\alpha_i = 2Y_i/Y_{\text{tot}}$. This and other special forms of scattering [5] are efficient computationally when admittances are real-valued, but in a general case it is practical to implement computation as shown in Fig. 1 so that the term $1/\sum Y_i$ is a common filter.

The freedom to use any impedance in a digital filter formulation allows also for applying numerical (e.g., measured) data as a part of a model. The condition for passivity is, as for a scattering junction in general, that Y_i must be positive real. The realization of junction nodes as shown in Fig. 1 is general for any LTI system approximation, also for 2-D and 3-D mesh structures. The delays (see W-line in Fig. 1) between nodes can also approximate fractional delays [11], which is useful particularly with varying length lines. However, delays shorter than a unit delay lead to the problem of delay-free loops, which complicates the situation substantially.

Dashed lines in Fig. 1 are parametric controls for admittances of the network elements. If the DSP blocks are grouped as shown, which is natural in an object-based formulation, the junction W-nodes actually contain most of the computation by implementing wave scattering. The W-node is delegated its admittance parameters through ports from the network elements, W-lines and W-admittances. In a time-varying case the admittance filters (blocks Y_1, Y_2, and Y_3) as well as the inverse of their sum $1/\sum Y_i$ must be updated when the admittance control parameters change.

2.2. Finite difference modeling

In the most commonly used way to discretize the wave equation by finite differences the partial derivatives in Eq. (1) are approximated by second order finite differences

$$y_{xx} \approx -(2y_{x,t} - y_{x-\Delta x,t} - y_{x+\Delta x,t})/(\Delta x)^2 \qquad (9)$$

$$y_{tt} \approx -(2y_{x,t} - y_{x,t-\Delta t} - y_{x,t+\Delta t})/(\Delta t)^2 \qquad (10)$$

By selecting the discrete-time sampling interval Δt to correspond to spatial sampling interval Δx, i.e., $\Delta t = c\Delta x$, and using index notation $k = x/\Delta x$ and $n = t/\Delta t$, Eqs. (9) and (10) result in

$$y_{k,n+1} = y_{k-1,n} + y_{k+1,n} - y_{k,n-1} \qquad (11)$$

which is a *K-model*, i.e., using Kirchhoff type of variables, not wave components. From form (11) we can see that a new sample $y_{k,n+1}$ at position k and time index $n+1$ is computed as the sum of its neighboring position values minus the value at the position itself one sample period earlier.

The behavioral similarity of DWGs vs. FDTDs [6], although being computationally different formulations (W- vs. K-models), hints to expand Eq. (11) to a FDTD type scattering junction for arbitrary port admittances. For a parallel admittance model, corresponding to Eq. (8), Eq. (11) can be formulated for N ports as

$$P_{J,n+1} = \frac{2}{Y_{\text{tot}}} \sum_{i=0}^{N-1} Y_i P_{i,n} - P_{J,n-1} \qquad (12)$$

This is the waveguide mesh formulationas discussed in [12]. Figure 2 depicts a DSP formulation of one such 3-port scattering *K-node* and the way to terminate port 1 by *K-admittance*, Y_1. This corresponds to the W-model in Fig. 1, except that a wave traveling to the left reflects back from Y_1 one unit delay later than in the DWG case. Notice the feedback through a unit delay. There can be any number of ports attached to a node also here as for a DWG junction. The bottom part of Fig. 2 depicts a block diagram abstraction which shows the conformity with the DWG in Fig. 1.

An essential difference between DWGs of Fig. 1 and FDTDs of Fig. 2 is that while DWG junctions are connected through two-directional delay lines (*W-lines*), FDTD nodes have two unit delays of internal memory, and delay-free *K-pipes* connect ports between nodes (see the right-hand side block in Fig. 2). The DWG and FDTD junction nodes and ports are not directly compatible because they use different type of wave variables.

The behavioral equivalence of the W- and K-models, i.e., DWG and FDTD models, can be shown simply by tracking a unit impulse wave component traveling in a waveguide consisting of different admittances, such as matched ones as well as short-circuited and open terminations. For the DWG case this is simple since the directional wave components are explicit. For a 1-D FDTD waveguide the equivalent (left-traveling) unit impulse at time moment n in node k is expressed by state $y_{k,n} = 1$, $y_{k+1,n-1} = 1$, and other states zero-valued, see Eq. (11). When the behavior is shown equivalent for an impulse wave component in both model types, it follows from the superposition principle that for any combinations of signals and network topologies the behavior of the W- and K-models is equivalent.

One further difference in K- vs. W-modeling, in addition to algorithmic and computational precision properties, is the possibility of 'spurious' responses in FDTD waveguides, i.e., an initial state of finite energy may generate waves of infinitely expanding energy in them [7]. For example, if the excitation U_{ext} (see Fig. 2) were injected directly to a node of an infinitely long 1-D FDTD waveguide, an oscillation of frequency $f_s/4$ starts to propagate from that point to both directions. To make only single impulses to start traveling, the input must be inserted through a filter $H(z) = 1 - z^{-2}$ where the term delayed by two unit samples cancels the inherent oscillatory behavior, making it equivalent to the injection in Fig. 1. The Nyquist frequency oscillation of FDTD is impossible in the DWG, but it can be simulated by feeding the injection through a recursive filter with poles at $z = \pm 1$.

One more notable 'nonphysical' property of the FDTD waveguide is its inherent integration ability, utilized in some modeling tasks [9, 1]. If an impulse excitation into a 1-D FDTD node is inserted through $H(z) = 1 + z^{-1}$, there will be step functions propagating to both directions from the excitation point. To do this with the DWG waveguide, a simple step function integrator $H(z) = 1/(1 - z^{-1})$ is needed for insertion. This integration property in an FDTD waveguide makes it possible to use a pair of variables, such as force and displacement, instead of the physically inherent pair of force and velocity.

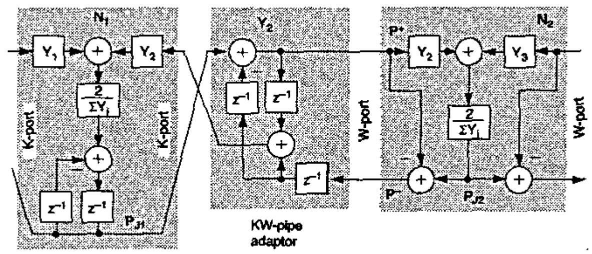

Figure 3: FDTD node (left) and a DWG node (right) forming a part of a hybrid waveguide. Y_i are wave admittances of W-lines, K-pipes, and adaptor KW-pipes between junction nodes. P_J are junction pressures, P^+ and P^- are wave components.

2.3. Interfacing DWGs and FDTDs

The next question is the possibility to interface wave-based and FDTD-based submodels. In [3] it was shown how to interconnect a lossy 1-D FDTD waveguide with a similar DWG waveguide into a mixed model using a proper interconnection element (adaptor). As a generalization, it is possible to make any hybrid model of K-elements (FDTD) and W-elements having arbitrary wave admittances/impedances at their ports.

Figure 3 shows how this can be done in a 1-D waveguide between a K-node N_1 (left) and a W-node N_2 (right). The role of the KW-pipe in the middle of Fig. 3 is to adapt the K-type port of an FDTD node and the W-type port of a DWG node. It is delay-free in left-to-right direction and contains delay in the opposite direction.

The desired functioning of the adaptor can be shown by testing the propagation of left- and a right-traveling impulses through the adaptor. The equivalence of W- and K-variable based models and the availability of the adaptor allow now to implement mixed models where both of the approaches can be applied, depending on which one is more useful in a problem at hand. Generally the DWG elements are preferable in 1-D modeling due to good numerical properties and possibility of arbitrary (including fractional) delays, while the FDTDs are more efficient in 2-D and 3-D structures, being however more critical in numerical accuracy.

2.4. Interfacing wave digital filters with DWGs

An addition to the mixed modeling framework above is to adopt Wave Digital Filters (WDF) [10] as discrete-time simulators of lumped parameter elements. Being based on W-modeling, i.e., wave variables, they are computationally compatible with the W-type DWGs [6]. A WDF *resistor* is basically just a real-valued termination (see Y_1) in Fig. 1, but WDF *capacitors* and *inductors*, as well as *ideal transformers* and *gyrators*, etc., are useful additional components [10].

As a physically bound choice for the case of this study, a WDF capacitor is realized as a feedback from V^- wave of a port back to V^+ through a unit delay, having a port admittance $2f_sC$. A WDF inductor is a feedback through a unit delay and coefficient -1, having a port admittance $1/2f_sL$. Here C is capacitance, L is inductance, and f_s is sample rate (cf. [6]).

Figure 4 shows an example of a model where a DWG delay line is terminated by a WDF capacitor at the left hand side and by a WDF inductor at the right hand side.

A beneficial property of WDF elements is, since their wave admittances are real-valued, that junctions of such ports remain memoryless in the sense of Fig. 1, i.e., Y_i and $1/\sum Y_i$ are real. On

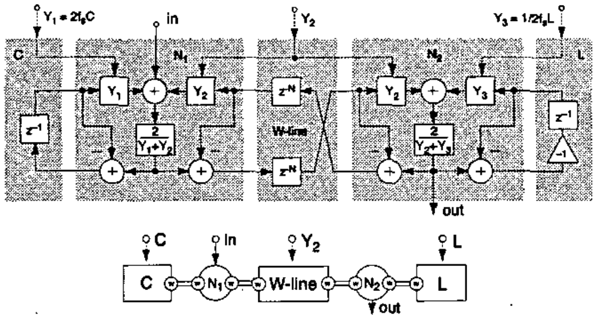

Figure 4: A simple DWG+WDF resonator where a DWG delay line is terminated with a WDF capacitor (left) and inductor (right).

the other hand, more flexibility and efficiency may be achieved in practice by higher order approximations of Y_i than by using basic WDF components. The WDF formulation helps more essentially in a problem that appears when nonlinearities or fast parametric changes in a system are to be modeled, where delay-free loops may appear, requiring special solutions such as [13].

WDF elements, being W-models, are not directly compatible with DFTDs that are K-models. However, the compatibility can be realized through a KW-adaptor element if needed.

3. REAL-TIME SIMULATION OF MIXED MODELS

The formulation of DWG + FDTD + WDF mixed models above shows directly a way to construct computational physical models based on DSP structures. A software system called the Block-Compiler has been developed that is particularly designed for most flexible yet efficient experimentation with physical models [4].

Object-oriented manipulation of blocks and their interactions in the BlockCompiler is implemented in the Common Lisp language. Automatic generation of C code from interconnected block representation and compilation to a run-time executable provides efficient computation in real time or a sample-by-sample non-real-time simulation. Multirate processing is available, where decimation or interpolation is specifiable for each block separately.

The basic level of blocks supports DSP with directed data flow, such as adders, multipliers, delays, and digital filters. A flexible macro block facility allows for abstractions, i.e., hiding low-level details in such macro blocks. Physical blocks with two-directional interaction through ports and junction nodes are built as macro block abstractions, characterized by shaded areas in Figs. 1–4. Utilizing DWG + FDTD + WDF blocks the BlockCompiler is developed toward a general discrete-time circuit and network simulator with parametric control of elements.

Scripting based on predefined DSP and physical blocks is relatively straightforward. DSP blocks are instantiated by make-forms such as `(.coeff 2.0)` for multiplier of 2.0, `(.ad)` for sound input and `(.da)` for sound output. These can be interconnected by an expression `(-> (.ad) (.coeff 2.0) (.da))` into a chain of output to input, output to input, etc. A full model is called a patch. It can be made streaming by `(run-patch my-patch)`. Physical block connections are made through ports as series or parallel connections, e.g., `(-s (port C) (port WL 0))` or simply `(-s C (port WL 0))` for a series connection of the capacitor and W-line on the left part of Fig. 4, from which evaluation the series junction node N_1 is resulted.

4. SUMMARY AND CONCLUSIONS

This paper has presented a DSP formulation of a physical modeling approach where several paradigms are combined to mixed models, particularly the digital waveguides (DWGs) and second-order finite difference time domain models (FDTDs). Furthermore, wave digital filters (WDFs) are available in addition to DWGs for wave-based modeling (W-modeling). FDTDs, based on Kirchhoff variables (K-modeling), are made compatible with W-models through an adaptor element. The formulation allows for high flexibility in building 1-D, 2-D, and 3-D physical models from interconnected blocks, supporting both spatially distributed and lumped elements. A software tool is developed where such models can be built and executed efficiently.

5. ACKNOWLEDGMENT

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