FREQUENCY-DEPENDENT SIGNAL WINDOWING

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ABSTRACT

Signal windowing is a temporal weighting operation whereby a signal is multiplied by a function that pays more emphasis on desired parts of the signal and typically attenuates it outside this span, normally to zero, in order to result in a finite support (nonzero part). In audio and acoustics applications it is quite common that better frequency resolution is desirable at low frequencies, while at high frequencies better time resolution calls for a shorter analysis or processing window. In this paper methods to realize and utilize frequency-dependent signal windowing are presented that exhibit this desirable frequency-dependent property. Particularly methods based on time-frequency warping and rewarping of signals are discussed. Audio examples are presented where this technique is found useful.

1. INTRODUCTION

Signal windowing is one of the most common operations in signal processing. It is motivated by the fact that finite support (span, range) is needed either due to limited processing capacity or most often due to non-stationary character of signals whereby they have to be processed frame by frame to obtain temporally localized representations.

Windowing in its traditional form is a multiplicative operation $(\cdot)$ in the time domain and thus corresponds to convolution $(\star)$ in the frequency domain, i.e.,

$$\begin{aligned} y(t) &= w(t)x(t) \\ Y(\omega) &= W(\omega) \star X(\omega) \end{aligned} \tag{1}$$

where $x(t)$ is a signal to be windowed, $w(t)$ is a windowing function, and $y(t)$ is the windowed signal. Upper case symbols denote Fourier transforms, respectively.

The form of a windowing function $w(t)$ can be any, but normally it is monotonically decreasing towards positive and negative time directions from its maximum value point, and for practical reasons achieves value 0 outside a specific span. Among most common symmetrical window functions are Hamming, Hann(ing), Blackman, Kaiser, and rectangular (or boxcar) window [1]. An asymmetrical window may be one-sided, such as an exponentially decaying (and truncated) window, or two-sided. The selection of window type and possible parameters associated with it depend on the criteria of each specific application at hand.

Traditional windows work in a frequency-independent manner. This is a straightforward solution, and with proper overlap-add or concatenation processing yields perfect reconstruction from consecutive windowed signal slices. In audio and acoustics applications, however, it is often useful to apply nonuniform time-frequency resolution. A well-known technique is the wavelet-based processing. In this paper we discuss other ways of controlling resolution, particularly methods that are based on time-frequency warping of signals and system responses. In addition to being a practical technique, it may help understanding the relation between warped techniques and wavelet-type processing.

Two basically different ways of frequency-dependent windowing are presented here: (a) frequency-domain and (b) time-domain techniques. The first one modifies the convolution rule of Eq. (1). The second one applies warping techniques, and it can be done either through frequency-domain resampling or through warped mappings. The main focus of this paper is on this last case.

2. GENERALIZED WINDOWING IN THE FREQUENCY DOMAIN

According to the frequency-domain version of windowing in Eq. (1), the Fourier transform of a windowed signal is obtained by (complex) convolution. When this is rewritten in the discrete Fourier transform case for frequency bin m by

$$Y(m) = \sum_{k=0}^{N} W_m(k) X(m-k) \tag{2}$$

where indices are modulo N, it becomes obvious that if we wish to obtain frequency-dependent windowing, the windowing term $W_m(k)$ cannot be a fixed vector but should be made bin-dependent. Each bin m can be treated separately by different 'window spreading' vector, designed according to the desired window for that specific bin. These vectors can be composed into an $N \times N$ matrix $\mathbf{W}$ for operation

$$\mathbf{y} = \mathbf{W}\mathbf{x} \tag{3}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the original and the windowed DFT vectors, respectively. Furthermore, the DFT and IDFT transforms can be formulated as matrix operations $\mathbf{q} = \mathbf{F}\mathbf{p}$ and $\mathbf{p} = \mathbf{G}\mathbf{q}$ where $\mathbf{p}$ is a signal vector, $\mathbf{q}$ is a spectrum vector, and $\mathbf{F}$ and $\mathbf{G}$ are the transform and its inverse transform composed of complex exponentials, respectively. Thus the whole chain of Fourier transform of signal $\mathbf{s}$, frequency-dependent windowing, and inverse transform to windowed signal $\mathbf{t}$, can be formulated as

$$\mathbf{t} = \mathbf{G}\mathbf{W}\mathbf{F}\mathbf{s} = \mathbf{M}\mathbf{s} \tag{4}$$

where $\mathbf{M}$ is a $N \times N$ matrix when the length of signal span $\mathbf{s}$ to be windowed is N and the frequency-domain window matrix $\mathbf{W}$ is also $N \times N$. For a traditional frequency-independent window $\mathbf{M}$ reduces to a diagonal matrix, i.e., to a sample-by-sample product of signal and window.

Due to the circularity of DFT and IDFT, frequency-dependent windowing may cause folding problems if not applied properly.

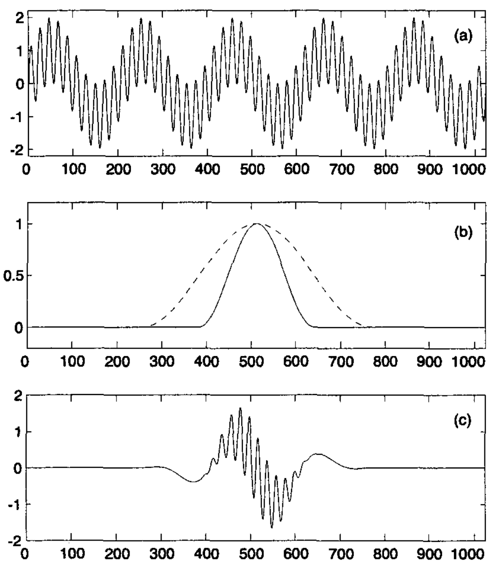

Figure 1: Frequency-dependent windowing of a sum of two sinusoids: (a) sum of sinusoids, (b) two Hann windows (solid line for high frequencies, dashed line for low frequencies), and (c) result of frequency-dependent windowing.

An example of using this method is given in Fig. 1 where the sum of two sinusoids is windowed so that a Hann window is shorter for the higher frequency than for the lower frequency.

3. GENERALIZED WINDOWING THROUGH TIME-FREQUENCY WARPING

The idea of *time-frequency warping* [2] can be characterized as follows. A sinusoid is scaled (expanded or compressed) in time by factor $\beta(f)$ depending on frequency f, which operation in the frequency domain corresponds to remapping of frequencies (resampling of frequency to keep uniform bin distribution):

$$A\sin(2\pi f[\beta(f)t] + \varphi) \leftrightarrow A\sin(2\pi[\beta(f)f]t + \varphi) \qquad (5)$$

Notice that warping is not a shift-invariant operation, thus the selection of time origin is of special importance. For impulse responses, time origin $t = 0$ is an inherently determined moment, but in the case of a general signal the selection of $t = 0$ specifies the only time moment where signal phase at all frequencies is not affected due to warping.

Time-frequency warping can be realized in two ways, (a) by frequency-domain resampling or (b) by analytical mappings. Frequency resampling [3] of signal $x(t)$ is realized by a sequence of operations

$$y(t) = \mathcal{F}^{-1}\{\mathcal{R}\{\mathcal{F}(x(t))\}\}$$

where $\mathcal{R}(\cdot)$ is a resampling operator, and $\mathcal{F}(\cdot)$ and $\mathcal{F}^{-1}(\cdot)$ are the Fourier transform and the inverse Fourier transforms, respectively. While frequency resampling yields a high degree of freedom for the warping function $\beta(f)$, the method is complicated and computationally expensive, and is not discussed further in this paper.

The second method of warping is based on transforms that map the complex z-domain unit disk onto itself. Each frequency value (point on the unit circle) will be mapped uniquely to a frequency value. There exists only one such rational function type, the bilinear conformal mapping [4]:

$$\tilde{z}^{-1} = D_1(z,\lambda) = \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}} \qquad (6)$$

where λ, $-1 < \lambda < 1$, is a warping parameter and $D_1(z,\lambda)$ is a dispersive delay element. This leads to a warping function [5]:

$$\beta(\Omega) = \frac{1}{\Omega}\arctan\frac{(1-\lambda^2)\sin(\Omega)}{(1+\lambda^2)\cos(\Omega) - 2\lambda}, \qquad (7)$$

where $\Omega = 2\pi f/f_s$, and f_s is the sampling frequency.

Mapping of a given impulse response $s(n)$ into warped impulse response $\tilde{s}(k)$ can be carried out simply based on equation

$$\sum_{k=0}^{\infty} \tilde{s}(k) z^{-k} = \sum_{n=0}^{\infty} s(n) D_1(z,-\lambda)^n \qquad (8)$$

and the mapping of $\tilde{s}(k)$ back into $s(n)$ based on equation

$$\sum_{n=0}^{\infty} s(n) z^{-n} = \sum_{k=0}^{\infty} \tilde{s}(k) D_1(z,\lambda)^k \qquad (9)$$

Mappings between sequences $s(n)$ and $\tilde{s}(k)$ are linear but not shift-invariant. Both mappings (8) and (9) may be computed with the same transversal warping structure but using coefficient λ for synthesis (9) and $-\lambda$ for analysis (8). This structure is similar to an FIR filter but with first-order allpass elements in place of unit delays, the impulse response to be mapped is used as tap coefficients, and the mapped response is achieved by feeding the structure with a unit impulse. Notice that both forms of (8) and (9) yield responses of infinite length even if the sequence to be mapped is of finite length, since allpass elements $D_1(z,\lambda)$ are internally recursive. For the selection of λ value, see for example [2].

Based on warping and inverse warping, *frequency-dependent windowing* is a technically easy task. The impulse response to be windowed is first warped according to Eq. (8) into the warped domain, then it is windowed properly as required for the application at hand, and finally inverse warped back according to Eq. (9). Notice that the inverse warped impulse response is theoretically infinite in length again, so that truncation or a secondary windowing is needed to make it finite.

For general signals for which the negative time sample values are not zero, warping can be implemented by warping separately for the positive time axis and backwards for the negative time axis, and finally concatenating the results.

4. EXAMPLE: WINDOWING OF IMPULSE RESPONSES OF ACOUSTIC SYSTEMS

Acoustic resonators and reverberating systems have often an impulse response with slower decay at low frequencies than at high frequencies. A measurement, such as an impulse response of a room or a musical instrument body, contaminated by acoustic or electronic noise, benefits from frequency-dependent windowing applied prior to further processing or modeling. When the effective window length is related to the inherent system response decay time, excessive noise at frequences where the response dies out fast is reduced correspondingly.

Figure 2 depicts the impulse response and magnitude response presentations for an acoustic guitar body, as measured by mechanical impulse hit at the bridge and registering the radiated acoustic

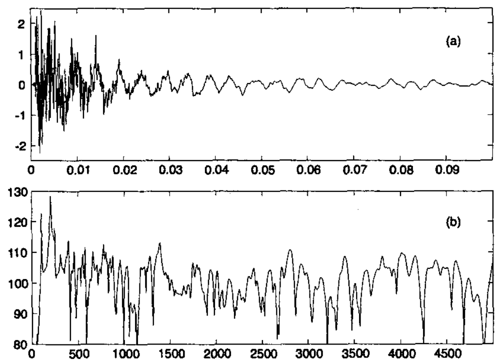

Figure 2: Measured response of an acoustic guitar body: (a) impulse response and (b) magnitude response.

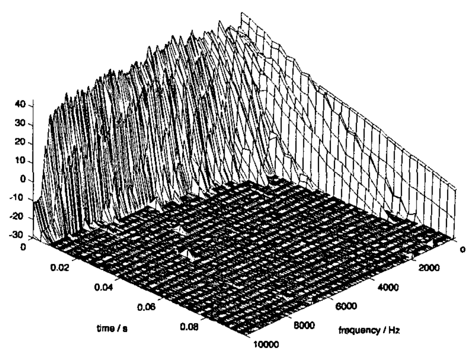

Figure 3: Time-frequency representation of the guitar body response.

response in front of the sound hole. Figure 3 illustrates the same response as a time-frequency plot (spectrogram). This shows clearly the difference in decay time between low and high frequencies. In the warped domain ($\lambda = 0.6$), the decay rate is balanced as shown in Fig. 4 so that all modes attenuate approximately equally fast.

If the measured (body) impulse response is more noisy, as depicted in Figs. 5a and 5b, warping and windowing can be utilized to improve the overall signal-to-noise ratio of the signal. Figures 5c and 5d demonstrate this by showing the result of applying a frequency-independent rectangular window (truncation) to the warped domain impulse response and synthesizing it back to the original time domain.

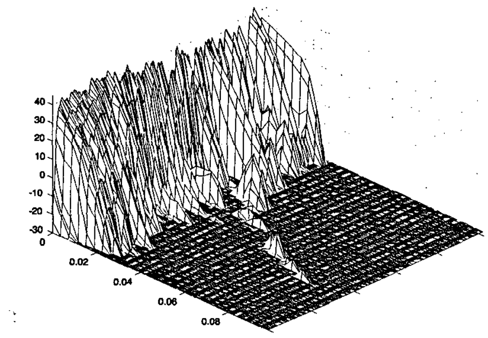

Figure 4: Time-frequency representation of the guitar body response in the warped domain.

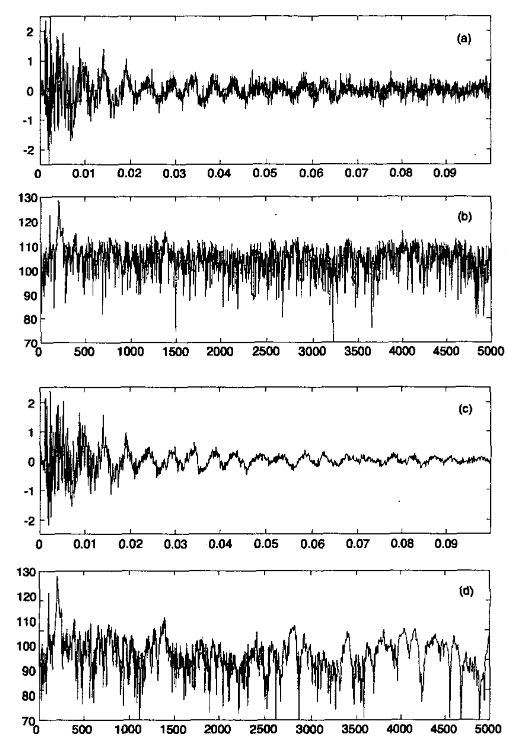

Figure 5: Noisy measurement of body response: (a) impulse response and (b) its magnitude spectrum. Enhanced version of the body response after frequency-dependent windowing through warping for noise reduction: (c) impulse response and (d) magnitude response.

5. APPLICABILITY OF KAUTZ STRUCTURES IN FREQUENCY-DEPENDENT WINDOWING

Kautz filters [7], or generalized transversal filters can be seen as an extension to the transversal warping structure where each transversal element may be different. For a given set of desired poles $\{z_i\}$ in the unit disk, the corresponding *rational orthonormal functions* are of the form

$$G_i(z) = \frac{\sqrt{1 - z_i z_i^*}}{z^{-1} - z_i^*} \prod_{j=0}^{i} \frac{z^{-1} - z_j^*}{1 - z_j z^{-1}}, \quad i = 0, 1, \ldots \qquad (10)$$

The Kautz filter is formed by replacing the kernel elements z^{-i} of an FIR filter with functions (10), which reduces to a transversal structure composed of a transversal all-pass backbone and all-pole tap-output filters. In general, the tap-output signals are complex valued, but for sets of real and complex conjugate poles we may

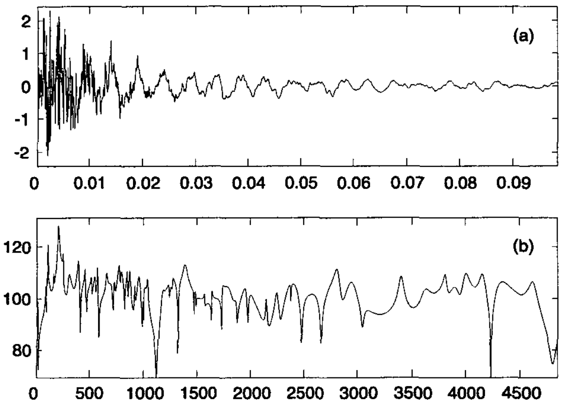

Figure 6: Kautz domain windowing enhanced guitar body response: (a) impulse response and (b) magnitude response.

use a modified real Kautz structure [6].

Kautz filters, applied as analysis structures, define orthonormal signal transformations $s(n) \rightarrow c(i)$ for any choice of poles $(z_i)_{i=0}^{\infty}$ in the unit disk. Due to the orthonormality, signal energy is preserved and any truncation $\hat{s}(n) = \sum_{i=0}^{N} c(i)g_i(n)$ in the Kautz domain defines a least-square optimal approximation with respect to the poles, i.e., a synthesis operation, where $\{g_i(n)\}$ are impulse responses or inverse z-transforms of functions (10). This is clearly a form of frequency-dependent windowing where desired frequency allocation and "feature extraction" is managed through the choice of poles and the (rectangular) windowing length in the Kautz domain.

We demonstrate the applicability of the proposed approach in noise reduction with two examples. In Fig. 6 we have chosen a Kautz domain window length 160 and optimized the pole positions with respect to the noisy response of Fig. 5a. Our pole optimization method is able to find quite well the true resonance structure (Fig. 2) from the noisy data, providing a significantly noise-reduced synthesized response. As another example we try to extract the sine wave from the noisy signal of Fig. 7a. The Kautz filter is chosen according to the (approximate) knowledge of the fundamental frequency. Displayed in Fig. 7b is the Kautz transformed signal from which we have chosen the window length 60 producing the synthesized signal of Fig. 7c. In this case of a Kautz structure with identical blocks we have a clearer interpretation of the windowing since the choice of poles and window length (modulo block size) is essentially separated.

6. SUMMARY AND CONCLUSIONS

This paper presented methods to apply frequency-dependent windowing to impulse responses and signals. The first formulation is specified in the frequency-domain and it can be realized as a time-domain matrix multiplication. The second method is to use first time-frequency warping, then conventional windowing, and finally inverse warping. A case study shows that frequency-dependent windowing can be used for example to enhance noisy impulse responses of acoustic systems.

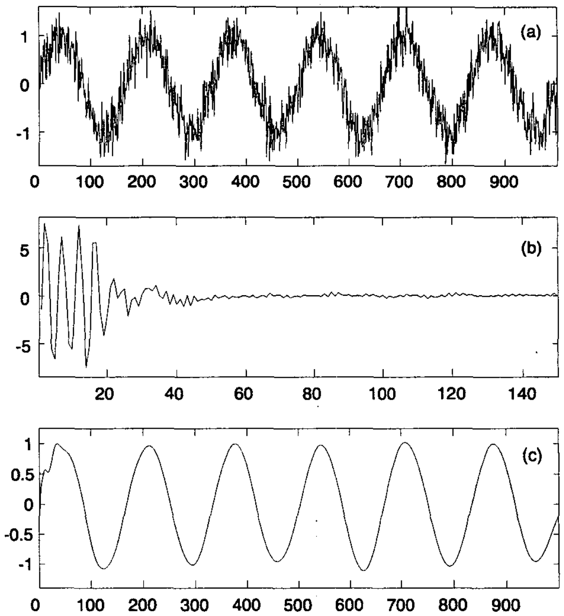

Figure 7: (a) Noisy sine wave, (b) its Kautz domain representation, and c) enhanced synthesized signal.

7. ACKNOWLEDGMENT

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8. REFERENCES

[1] T. Saramäki, "Finite impulse response filter design," Ch. 4 in *Handbook for Digital Signal Processing*, ed. S. K. Mitra and J. F. Kaiser, Wiley 1993.

[2] A. Härmä, M. Karjalainen, L. Savioja, V. Välimäki, U. K. Laine, and J. Huopaniemi, "Frequency-warped signal processing for audio applications," *J. Audio Eng. Soc.*, vol. 48, no. 11, pp. 1011-1031, Nov. 2000.

[3] J. O. Smith, *Techniques for digital filter design and system identification with application to the violin*, Ph.D. thesis, Elec. Eng. Dept., Stanford University, 1983.

[4] Churchill R. V., *Complex Variables and Applications*. McGraw-Hill, New York, 1960.

[5] M. Karjalainen, A. Härmä, U. K. Laine, and J. Huopaniemi, "Warped Filters and Their Audio Applications," *Proc. WASPAA-97*, Mohonk, New Paltz, 1997.

[6] P. W. Broome, "Discrete Orthonormal Sequences", *Journal of the Association for Computing Machinery*, vol. 12, no. 2, pp. 151–168, 1965.

[7] W. H. Kautz, "Transient Synthesis in the Time Domain", *IRE Trans. Circuit Theory*, vol. CT–1, pp. 29–39, 1954.