Kautz Filters and Generalized Frequency Resolution: Theory and Audio Applications*

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Frequency-warped filters have recently been applied successfully to a number of audio applications. The idea of all-pass delay elements replacing unit delays in digital filters allows for focusing enhanced frequency resolution on the lowest (or highest) frequencies and enables a good match to the psychoacoustical Bark scale. Kautz filters can be seen as a further generalization, where each all-pass element may be different, allowing also complex-conjugate poles. This enables an arbitrary allocation of frequency resolution to filter design, such as modeling and equalization (inverse modeling) of linear systems. Strategies for using Kautz filters in audio applications are formulated. Case studies of loudspeaker equalization, room response modeling, and guitar body modeling for sound synthesis are presented.

0 INTRODUCTION

In the context of signal processing, rational orthonormal filter structures were first introduced in the 1950s by Kautz, Huggins, and Young [1]–[3]. Kautz showed that an orthogonalization process applied to a set of continuous-time exponential components produces orthonormal basis functions having particular frequency-domain expressions. Much earlier Wiener and Lee [4] proposed synthesis networks based on some classical orthonormal polynomial expansions [5]. The idea of representing functions in orthonormal components is elementary in Fourier analysis, but the essential observation in the aforementioned cases was that some time-domain basis functions have rational Laplace transforms with a recurrent structure, defining an efficient transversal synthesis filter.

Discrete-time rational orthonormal filter structures can be attributed to Broome [6] as well as the baptizing of the discrete Kautz functions, consequently defining the discrete-time Kautz filter. The point of reference in the mathematical literature is somewhat arbitrary, but reasonable choices are the deductions made in the 1920s to prove interconnections between rational approximations and interpolations, and the least-square (LS) problem, which were assembled and further developed by Walsh [7].

Over the last ten years there has been a renewed interest toward rational orthonormal model structures, mainly from the system identification point of view [8]–[11]. The perspective has usually been to form generalizations to the well-established Laguerre models in system identification [12]–[14] and control [15]. In this context the Kautz filter or model has often the meaning of a two-pole generalization of the Laguerre structure [9], whereupon further generalizations restrict as well to structures with identical blocks [16]. Another, and almost unrecognized, connection to recently active topics in signal processing are the orthonormal state-space models for adaptive IIR filtering [17], with some existing implications to Kautz filters [18], [19].

Kautz filters have found very little use in audio signal processing, to cite rarities [20]–[24]. One of the reasons is certainly that the field is dominated by the system identification perspective. The more inherent reason is that there is an independent but related tradition of frequency-warped structures, which is already well grounded and sufficient for many tasks in audio signal processing. Frequency warping provides a (rough) approximation of the constant-$Q$ resolution of modeling [25] as well as a good match with the Bark scale, which is used to describe the psychoacoustical frequency scale of human hearing [26]. For a recent overview of frequency warping, see [27].

In our opinion there is a certain void in generality, both in the proposed utilizations of Kautz filters as well as in the warping-based view of the frequency resolution of modeling. There are some implications [28], [29] that Kautz filters would also be well applicable to adaptive modeling of audio systems, such as acoustic echo cancel-
lation and channel equalization. In this paper we demonstrate the use of Kautz filters in pure filter synthesis, that is, modeling of a given target response, which was actually their original usage. Quite surprisingly, to our knowledge there have not actually been any proposals in this direction on the level of modern computational means in design and implementation.

The objective of this paper is to introduce the concept of Kautz filters in a practical manner and from the point of view of audio signal processing. After a brief theoretical part we present various methods for the most essential ingredient in Kautz filter design, namely, choosing a particular structure. Audio applications of Kautz filtering, including loudspeaker equalization, room response modeling, and modeling of an acoustic guitar body, are demonstrated and compared with more traditional approaches.

1 KAUTZ FILTER STRUCTURES

For a given set of desired poles \( \{z_i\} \) in the unit disk, the corresponding set of rational orthonormal functions is uniquely defined in the sense that the lowest order rational functions \( G_i(z) \), square-integrable and orthonormal on the unit circle, analytic for \( |z| > 1 \), are of the form [7]

\[
G_i(z) = \prod_{j=0}^{i-1} \frac{z^{-1} - z_j^*}{1 - z_j z_i^{-1}}, \quad i = 0, 1, \ldots (1)
\]

where \( z^* \) denotes the complex conjugate of \( z \). The meaning of orthonormality is economically established in the time domain: for the impulse responses \( g_i(n) \) of Eq. (1), the time-domain inner products satisfy

\[
(g_i, g_k) = \sum_{n=0}^{\infty} g_i(n) g_k^*(n) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \quad (2)
\]

For the remaining conditions of square integrability and analyticity it is sufficient to presume stability and causality of the rational transfer function, that is, that \( |z| < 1 \) for all the poles.

Eq. (1) forms clearly a repetitive structure: up to a given order, that is, the number of poles, functions associated with the subsets \( \{z_i\}_{j=0}^{i} \) of an ordered pole set \( \{z_i\}_{j=0}^{N} \) are produced as intermediate substructures defining a tapped transversal system. In agreement with the continuous-time counterpart, a weighted sum of these functions is called a Kautz filter and depicted in Fig. 1.

Defined in this manner, Kautz filters are merely a class of fixed-pole IIR filters, forced to produce orthonormal tap-output impulse responses. A particular Kautz filter is thus determined by a pole set, consequently defining the filter order, the all-pass filter backbone, as well as the corresponding tap-output filters. The Kautz filter response is thus defined by the linear-in-parameters \( \{w_i\} \) formulas

\[
H(z) = \sum_{i=0}^{N} w_i G_i(z) \quad \text{and} \quad h(n) = \sum_{i=0}^{N} w_i g_i(n) \quad (3)
\]

which despite their appearance as parallel systems are transversal in nature because of the particular form of Eq. (1).

A gentle approach to understanding Kautz filters from a traditional digital filtering point of view is to start from an FIR filter and replace the unit delays by first-order all-pass sections, that is, by frequency-dependent delays. If the sections are equal, the resulting structure is a warped FIR filter (WFIR) [27]. With an additional cascaded normalization term the structure is a Laguerre filter (normalized warped filter). Finally, when the all-pass sections have individual pole values, possibly complex ones, the result is a Kautz filter, a generalized transversal filter. In analogy to the FIR filter, Eq. (3) can be seen as a truncated generalized \( z \) transform or as a generalization of the unit-delay decompositions of a given system. Note that due to internal feedbacks it is a filter with an infinite impulse response.

Fig. 2 illustrates time- and frequency-domain responses of an example Kautz filter with a real pole set \( \{0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9\} \). In the time domain the tap outputs have excessively increasing delays, similar to FIR filter taps but with infinitely long responses. In the frequency domain each tap has a magnitude response that is independent of the tap position, in this sense resembling a parallel filter bank.

1.1 Kautz Modeling of Signals and Systems

Defined by any set of points \( \{z_i\}_{j=0}^{N} \) in the unit disk, Eq. (1) forms an orthonormal set that is complete, or a base, with a moderate restriction on the poles \( \{z_i\} \). The corresponding time-domain basis functions \( \{g_i(n)\}_{j=0}^{N} \) are impulse responses or inverse \( z \) transforms of Eq. (1). This implies that a basis representation of any causal and fini-

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1Kautz filters are clearly related to other orthogonal filter formulations, but here we just exploit this connection by referring to favorable numerical properties implied by the orthonormality [30]–[33].
energy discrete-time signal is obtained as its generalized
Fourier series, that is, as infinite sums of the form of Eqs. (3) with

\[ \sum_{n=0}^{\infty} |h(n)| < \infty \Rightarrow \sum_{n=0}^{\infty} |h(n)|^2 < \infty \] (4)

that is, impulse responses of causal and stable (CS) linear
time-invariant (LTI) systems form a subset of finite energy
signals, which makes Eqs. (3) valid model structures for
input-output data identification.

Kautz filters provide models for many types of system
identification and approximation schemes, including
adaptive filtering, for both fixed and nonfixed pole struc-
tures. Also in the Kautz model case there are various inter-
pretations, criteria, and methods for the parameterization
of the model. Here we address only the “prototype” LS
approach implied by the signal space description: tap-
output signals of a Kautz filter \( x_i(n) = G_i[x(n)] \) to input
\( x(n) \) span a finite-dimensional approximation space, pro-
viding an LS optimal approximation to any CSLTI system
with respect to the basis. The parameterization, that is, the
filter weights are solutions of the normal equations assembled
from correlation terms of the tap outputs and the
desired output \( y(n) \). For example, in matrix form, defining
the correlation matrix \( R \), \( r_{ij} = (x_i, x_j) \), the weight vector
\( w \) is the solution of the matrix equation

\[ Rw = p \quad [p]_i = (y, x_i) \] (5)

where \( p \) is the correlation vector. This simply utilizes the
previously defined time-domain inner product Eq. (2) of
the signals, so there are also definitions for the frequency-
domain correlation terms, and separately for deterministic
and stochastic signal descriptions, but this notion is just
included to indicate similarities with the FIR modeling
case.

Actually we further limit our attention to the approxi-
mation of a given target response since the input-output
system identification framework is a bit pompous and
impractical for most audio signal processing tasks. Although typical operations such as inverse modeling and
equalization are basically identification schemes, they are
usually convertible to the approximation problem. Furthermore we are not going to address here the interest-
 ing question of the invertibility of a Kautz filter, which
would require elimination of delay-free loops, or an imple-
mentation method proposed in [34].

For a given system \( h(n) \) or \( H(z) \), Fourier coefficients
provide LS optimal parameterizations for the correspond-
ing Kautz model, or synthesis filter, with respect to the
pole set. Evaluation of the Fourier coefficients

\[ c_i = \langle h, g_i \rangle = \langle H, G_i \rangle \] (6)

can be implemented by feeding the time-reversed signal
\( h(-n) \) to the Kautz filter and reading the tap outputs \( x_i \)
at time \( n = 0 \),

\[ c_i = x_i(0), \quad x_i(n) = h(-n) * g_i(n). \] (7)

This implements inner products by filtering (convolution
\([\ast]\), and it can be seen as a generalization of the FIR
design by truncation. It should be noted that here the LS
 criterion is applied on the infinite time horizon and not, for
example, in the time window defined by \( h(n) \). We use these
two orthonormal expansion coefficients because they are
easy to obtain, providing implicitly simultaneous time- and
frequency-domain design and powerful means to the Kautz
filter structure (that is, pole position) optimization. More-

\[ 2 \text{The term correlation is used to point out that this is a genuine}
generalization of the Wiener filter setup.}
over, the coefficients are independent of ordering and approximation order, which makes choosing poles, approximation error evaluation, and model reduction efficient.

1.2 Real-Valued Kautz Functions for Complex-Conjugate Poles

A Kautz filter produces real tap-out signals only in the case of real poles. In principle this does not in any way limit its potential capabilities of approximation a real signal or system. However, we may want the processing, that is, signals internal to a filter, coefficients, and arithmetic operations to be real-valued. A restriction to real linear-in-parameter models can also be seen as a categoric step in the optimization of the structure.

From a sequence of real or complex-conjugate poles it is always possible to form real orthonormal structures. Symbolically this is done by applying a block diagonal unitary transformation to the outputs, consisting of 1’s corresponding to real poles and 2 × 2 rotation elements corresponding to pole pairs. From the infinite variety of unitarily equivalent possible solutions it is sufficient to use the intuitively simple structure of Fig. 3, proposed by Broome [6].

The second-order section outputs of the backbone structure in Fig. 3 are already orthogonal, from which tap-output pairs are produced as orthogonal components of difference and sum, x(n) − x(n − 1) and x(n) + x(n − 1), respectively. The orthonormalizing terms are then

\[ p_i(z^{-1} - 1) \quad \text{and} \quad q_i(z^{-1} + 1) \quad (8) \]

and they are determined by the corresponding pole pair \( \{z_i, z_{i}^{*}\} \) so that

\[ p_i = \sqrt{(1 - \rho_i)(1 + \rho_i - \gamma_i)} / 2 \]
\[ q_i = \sqrt{(1 - \rho_i)(1 + \rho_i + \gamma_i)} / 2 \]
\[ \gamma_i = -2 \text{Re} \{z_i\} \]
\[ \rho_i = |z_i|^2 \]

where \( \gamma_i \) and \( \rho_i \) can be recognized as corresponding second-order polynomial coefficients. The construction works also for real poles, producing a tap-output pair corresponding to a real double pole, but we use a mixture of first- and second-order sections in the case of both real and complex-conjugate poles.

A Kautz filter example with complex-conjugate pole pairs is characterized by frequency-domain responses in Fig. 4(b) and (c). Notice the complementary orthogonal behavior of resonances for odd versus even tap outputs. Kautz pole positions in the \( z \) plane are shown by circles in Fig. 4(a).

2 METHODS FOR POLE DETERMINATION

Contrary to all-pole or all-zero modeling and filter design there are in general no analytical methods to solve for optimal pole–zero filter coefficients. Kautz filter design can be seen as a two-step procedure, involving the choosing of a particular Kautz filter pole set and the evaluation of the corresponding filter weights. The fact that the latter task is much easier, better defined, and inherently LS optimal makes it tempting to use sophisticated guesses and random or iterative search in the pole optimization. For a more analytic approach, the whole idea in the Kautz concept is how to incorporate desired a priori information to the Kautz filter. This may mean knowledge on system poles or resonant frequencies and the corresponding time constants, or indirect means, such as any all-pole or pole–zero modeling method to find potential Kautz filter poles.

A practical way to limit the degrees of “freedom” in filter design is to restrict to structures with identical blocks,

\[ \text{According to Eq. (5), coefficients } c_i \text{ depend only on the corresponding basis function. This means that for a fixed (ordered) base } \{g_i\}_{i=0}^{\infty} \text{ we obtain the same coefficients for any chosen approximation order. Moreover, the energy of the approximation is distributed orthogonally in the coefficients, which makes the approximation error } E = (h, h) - \Sigma_{i=0}^{\infty} |c_i|^2, \text{ independent of the ordering of the basis functions. For various permutations of a fixed basis function set, } \{g_i\}_{i=0}^{\infty} \text{ the tap-output magnitude responses are indeed independent of ordering, but the phase responses differ. However, the phase response of the approximation } h(n) = \Sigma_{i=0}^{\infty} c_i g_i(n) \text{ is independent of ordering.}

\[ \text{Assuming that we are approximating a real-valued response, we know that the "true" poles of the system are real or occur in complex-conjugate pairs, which makes all other choices suboptimal.}

\[ \text{Some of the Kautz filter deductions are made directly on the real function assumption [6]. Moreover, the state-space approach to orthonormal structures with identical blocks, the generalized orthonormal basis functions of Heuberger, is based on balanced realizations of real rational all-pass functions [8].} \]
that is, to use the same smaller set of poles repeatedly. The pole optimization and the model order selection problems are then essentially separated, and various optimization methods can be applied to the substructure. In addition, for the structure of identical blocks, a relation between optimal model parameters and error energy surface stationary points with respect to the poles may be utilized [35] as well as a classification of systems to associate systems and basis functions [36]. However, in the following we mainly focus on practical methods for choosing a distributed pole set.

2.1 Generalized Descriptions of Frequency Resolution

As mentioned earlier, frequency-warped configurations in audio signal processing [27] constitute a self-contained tradition originating from warping effects observed in analog-to-digital mappings and digital filter transformations [37]. The concept of a warped signal was introduced to compute nonuniform resolution Fourier transforms using the fast Fourier transform (FFT) [25] and in a slightly different form to compute warped autocorrelation terms for warped linear predictions [38]. The original idea of replacing a unit delay element with a first-order all-pass operator in a transfer function, that is,

$$z^{-1} \mapsto \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}}$$

was restated and applied to general linear filter structures [39]–[41].

The warping effect or resolution description introduced by first-order all-pass warping is defined by the all-pass

![Diagram of real-valued Kautz filter for complex-conjugate pole pairs.](image)
phase function of Eq. (10). The warping parameter $\lambda$ can be chosen to approximate desired frequency-scale mappings, such as the perceptually motivated Bark scale, with respect to different error criteria and sampling rates [26]. This is complete in the sense that the first-order all-pass element is the only (rational, stable, and causal) filter having a one-to-one phase function mapping. Parallel structures can be constructed to approximate any kind of warping [42], including the logarithmic scale [43], but the aim of this chapter is to broaden the concept of frequency resolution to account for the resolution allocation introduced by a Kautz filter.

There is a certain deep-rooted conceptual difficulty in placing Kautz filter techniques in the general framework of LTI filtering and its warped counterparts. Warped filters provide an efficient way to incorporate a desired frequency resolution in traditional FIR and IIR filter design, if the bilinear mapping in Eq. (10) is considered sufficient and flexible enough. However, a more fine-structured or case-specific frequency resolution allocation would often be preferable. On the other hand, it can be theoretically proven that FIR filters are the optimal choice if (exponential) stability is the only thing that is known of the system to be modeled. Actually the opposite of this statement is used in motivation of the Kautz model for approximating resonant systems [36]. Furthermore, in our cases of typical audio-related responses it can be readily shown that the Kautz filter techniques proposed in Section 2.3 usually provide in the LS sense the best IIR filter designs, which may seem surprising regarding the vast number of methods available. Warped counterparts of FIR and IIR filters might be the optimal choices in producing a desired simple global allocation of frequency resolution, but for a specific filter design problem this is probably not the case. Combining these arguments, our hypothesis is that if a target response has clearly distinguishable resonant features, it is always possible to design a Kautz filter with at least the same technical and perceptual quality, but with better efficiency.

The Laguerre filter differs structurally from the warped FIR filter only in the orthonormalizing all-pole prefilter,

$$H(z) = \frac{\sqrt{1-\lambda^2}}{1-\lambda z^{-1}} \sum_{i=0}^{N} w_i \left( 1 - \frac{1}{1-\lambda z^{-1}} \right)^i$$

(11)

that is, the term before the weighted tap summation (see also Fig. 1). The effect of this distinction can be seen as a mere technicality, which can be compensated if needed, or even as a favorable frequency emphasis in the modeling [38]. Actually we have already referred to three variations of warping, the Laguerre and the all-pass warping and the Oppenheim et al. construction [25], which is a biorthogonal analysis–synthesis structure. In practice the question which warping to choose may not be of importance, but signal and system transformations defined by the Laguerre analysis–synthesis structure are the only truly orthonormal transform pairs related to mapping (10).

The orthonormal Laguerre transformation of signals and systems has a generalization to Kautz structures with identical blocks defined by any rational all-pass transformation [44], [10], so actually it is just a question of how to interpret the well-defined phase characteristics as a frequency resolution mapping. The same applies to the general Kautz model. The phase function of the all-pass backbone is completely determined by the pole set (as a whole) and we should “just” find a way to decode this information as a generalized frequency resolution allocation. Audio application cases presented in Section 3 help with understanding intuitively how to focus resolution on the frequency axis in a desirable way.

Fig. 5 also characterizes this from a couple of viewpoints for the same case of complex-conjugate poles with logarithmically distributed pole angles and constant pole radius, as was used in Fig. 4. Fig. 5(b) shows the phase functions for each individual pole-pair section and Fig. 5(c) plots the accumulated phases at the tap outputs of the all-pass sections (from high-frequency to low-frequency pole angles). Fig. 5(d) is the group delay $T_g = -d\phi/d\omega$ of the all-pass backbone. The accumulated phase in Fig. 5(c) is equivalent to the effect of frequency warping in the Laguerre case, although the concept of warping is not as clear and should be used with caution. Similarly, the group delay can be interpreted as a measure of frequency resolution. It illustrates well how the resolution in this case is highest at low frequencies and then decreases, except locally at frequencies corresponding to the resonating pole pairs.

A trivial way to attain a desired frequency resolution allocation in Kautz modeling is to use suitable pole distributions. We may place (complex-conjugate) poles with pole angle spacings corresponding to any frequency resolution mapping. The choosing of pole radii is also to be taken into account so that we can use more sophisticated choices than a constant radius. Motivated from our experiments on producing warping on a logarithmic scale with parallel all-pass and generalized parallel orthonormal structures [43] we have, for example, used pole radii inversely proportional to the pole angles.

### 2.2 Manual Fitting to a Given Response

It is always possible to simply adjust the Kautz filter poles manually through trial and error in order to produce a Kautz filter matching well with a given target response. A Kautz filter impulse response is a weighted superposition of damped sinusoids which provide for direct tuning of a set of resonant frequencies and the corresponding decay time constants. In principle this allows for very flexible time-domain design, especially if some other weighting than the LS parameterization is chosen.

By direct inspection of the time and frequency responses it is relatively easy to find useful pole sets by selecting a set of prominent resonances and proper pole radius tuning. Choosing the complex-conjugate pole angles is more critical than the pole radius selection in the sense that the filter coefficients perform automatic weighting of the sinusoidal components. In practice, however, when designing low-order models for highly resonant systems, poles must be fine-tuned very close to the unit circle.

An obvious way to improve the overall modeling with a structure based on a fixed set of resonances is to use the corresponding generating substructure repetitively, produc-
ing a Kautz filter with identical blocks. There is no obligation to use the same multiplicity for all poles, but it makes model reduction easier. If a set of poles is determined for a substructure that is used repetitively, some kind of damping by reduced pole radii would be advisable.

2.3 Methods Implied by Orthogonality

We have chosen our model to a given target response \( h(n) \) to be the truncated series expansion

\[
\hat{h}(n) = \sum_{i=0}^{N} c_i g_i(n), \quad c_i = (h, g_i)
\]

(12)

which makes approximation error evaluation and control easy. For any (orthonormal) \( \{g_i\} \) and approximation order \( N \), the approximation error energy satisfies

\[
E = \sum_{i=N+1}^{\infty} |c_i|^2 = H - \sum_{i=0}^{N} |c_i|^2
\]

(13)

where \( H = (h, h) \) is the energy of \( h(n) \). Hence we get the energy of an infinite-duration error signal as a by-product from finite filtering operations, previously described for the evaluation of the filter weights.

As a more profound consequence of the orthogonality, the all-pass operator defined by the (transversal part of the) chosen Kautz filter induces a complementary division of the energy of the signal \( h(-n), n = 0, \ldots, M [3] \). An all-pass filter \( A(z) \) is lossless by definition, and (from the preceding) it can be deduced that the portion of the response \( a(n) = A[h(-n)] \) in the time interval \([-M, 0]\) corresponds to the approximation error \( E \), that is, the Kautz filter optimization problem reduces to the minimization of the energy of a finite-duration signal.

We have encountered three attempts to utilize the concept of complementary signals in the pole position optimization, at least implicitly [45]–[47]. McDonough and Huggins replace the all-pass numerator with a polynomial approximating the denominator mirror polynomial to produce linear equations for the polynomial coefficients in an iteration scheme [45]. Friedman has constructed a network structure for parallel calculations of all partial derivatives of the approximation error with respect to the real second-order polynomial coefficient associated to the complex pole pairs, to be used in a gradient algorithm [46].

Brandenstein and Unbehauen have proposed an iterative method resembling the Steiglitz–McBride method of pole–zero modeling to pure FIR-to-IIR filter conversion [47], which we have adapted to the context of Kautz filter optimization. We have implemented and experimented with all the aforementioned algorithms, but in general just the method of Brandenstein and Unbehauen (BU method) is found reliable. It genuinely optimizes in the LS sense the pole positions of a real Kautz filter, producing unconditionally stable and (theoretically globally) optimal pole sets for a desired filter order. Furthermore the BU method works on relatively high filter orders, for example, providing sets of 300 distributed and accurate poles, which would be unachievable with standard pole–zero modeling methods.

\[6\] An iterative solution based on a modified error criterion is presented in [48] as well as a connection to the generalized Steiglitz–McBride method in adaptive IIR filtering in [17].
An outline of our Kautz filter pole-generation method, modified from Brandenstein and Unbehauen [47] follows.

The algorithm is based on approximating an all-pass operator \( A(z) \) of a given order \( N \) with

\[
A^{(k)}(z) = \frac{z^{-N} D^{(k)}(z^{-1})}{D^{N-1}(z)}
\]

(14)

where \( \{1, D^{(1)}(z), D^{(2)}(z), \ldots \} \) are iteratively generated polynomials, restricted to the form

\[
D^{(k)}(z) = 1 + \sum_{i=1}^{N} d_i^k z^{-i} = 1 + D_1^{(k)}(z), \quad k = 1, 2, \ldots
\]

(15)

Eq. (14) converges to an all-pass function if \( ||D^{(k)}(z) - D^{(k-1)}(z)|| \to 0 \).

The objective is to minimize the output of Eq. (14) to the input \( X(z) = z^{-L} H(z^{-1}) \) [the \( z \)-transform of \( h(-n) \)]. Denote \( Y^{(k)}(z) = A^{(k)}[X(z)] \).

Define \( Y^{(k)}(z) = X(z)D^{(k-1)}(z) \) [all-pole filtered \( h(-n) \)].

Now \( U^{(k)}(z) = z^{-N} D^{(k)}(z^{-1})Y^{(k)}(z) \), and by substituting Eq. (15) and rearranging,

\[
Y^{(k)}(z) z^{-(N-1)} D_1^{(k)}(z^{-1}) = U^{(k)}(z) - z^{-N} Y^{(k)}(z).
\]

(16)

Collecting common polynomial terms into a matrix equation produces \( A^{(k)} \delta^{(k)} = u^{(k)} + b^{(k)} \), where \( \delta^{(k)} \) and \( u^{(k)} \) are unknown. The solution of \( A^{(k)} \delta^{(k)} = b^{(k)} \) minimizes the square norm of \( u^{(k)} = A^{(k)} \delta^{(k)} - b^{(k)} \).

The BU algorithm proceeds as follows:

Step 1: For \( k = 1, 2, \ldots \), filter \( h(-n) \) by \( 1/D^{(k-1)}(z) \) to produce the elements of \( A^{(k)} \) and \( b^{(k)} \).

Step 2: Solve \( A^{(k)} \delta^{(k)} = b^{(k)} \), \( \delta^{(k)} = A^{(k)} \delta^{(k)} - b^{(k)} \), the (mirror) polynomial coefficients of \( D_1^{(k)}(z) \). Go to step 1.

Step 3: From a sufficient number of iterations, choose \( D^{(k)}(z) \) that minimizes the true LS error. The Kautz filter poles are the roots of \( D^{(k)}(z) \).

2.4 Hybrid Methods

By trying to categorize various methods for choosing a particular Kautz filter we do not in any way intend to be complete or exclusive. There are certainly many other possibilities as well as modifications and mixtures of the ones presented. For example, we have not addressed pole position optimization methods based on input–output descriptions of the system, which could in some cases be useful even if the target response is available.

An obvious way to modify the optimization is to manipulate the target response. Time-domain windowing or frequency-domain weighting by suitable filtering may be applied, or alternatively, the optimization can be partitioned using selective filters. For example, the BU method is based on solving at every iteration a matrix equation with the dimensions implied by the duration of the target response and the chosen model order, and therefore pathological behavior can be decreased by truncation. By divid-
Kautz model magnitude response is due to the choice of poles, and it is not a property of the model, in contrast to some warped designs. A simplified version of the proposed method is to down-sample the response prior to the BU method and to map the produced poles back to the original frequency domain. This approach is used in case 2.

3 AUDIO APPLICATION CASES

We have tested the applicability of Kautz filter design in three audio-oriented applications. The first is a loudspeaker equalization task where frequency resolution is distributed both globally and locally. The second case, a room impulse response, is modeled at low frequencies to capture the modal behavior as a robust filter structure. In the third case we use Kautz filters to model the body response of an acoustic guitar where, similarly, the lowest frequencies are of primary interest.

3.1 Case 1: Loudspeaker Response Equalization

An ideal loudspeaker has a flat magnitude response and a constant group delay. Simultaneous magnitude and phase equalization of a nonperfect loudspeaker would be achieved by modeling the response and inverting the model, or by identifying the overall system of the response and the Kautz equalizer, but here we demonstrate the use of Kautz filters in pure magnitude equalization based on an inverted target response.

A small two-way active loudspeaker was selected as a Kautz equalization experiment due to its clear deviations from ideal magnitude response. The measured response and a derived equalizer target response are included in Fig. 6. The sample rate is 48 kHz.

Magnitude response equalization consists typically of compensating for three different types of phenomena: 1) slow trends in the response, 2) sharp and local deviations, and 3) correction of rolloffs at the band edges. This makes “blind equalization” methods, which do not utilize audio-specific knowledge, ineffective. We propose that Kautz filters provide a useful alternative between blind and hand-tuned parametric equalization.

As is well known, FIR modeling and equalization has an inherent emphasis on high frequencies on the auditorily motivated logarithmic frequency scale. Warped FIR (or Laguerre) filters [27] shift some of the resolution to the lower frequencies, providing a competitive performance with 5 to 10 times lower filter orders than with FIR filters [49]. However, the filter order required to flatten the peaks at 1 kHz in our example is still high, on the order of 200, and in practice Laguerre models up to the order of about 50 are able to model only slow trends in the response. In a recent publication [50] general real-pole all-pass cascades were proposed for low-frequency equalization, but they were found difficult to design. Here we demonstrate efficient design methods for the orthonormal and complex-pole counterpart.

The simplest choice of Kautz filter poles in this case is to focus on the 1-kHz region with a single (or multiple) pole pair. By tuning the pole radius we trade off between the 1-kHz region and the overall modeling. Quite interestingly, as a good compromise, we end up with a radius close to a typical warping parameter at this sampling rate (for example, \( \lambda = 0.76 \)) [26] and we get surprisingly similar results for the Laguerre and the two-pole Kautz equalizers for filter orders 50–200. Actually this simply means that a perceptually motivated warping is also technically a good choice for the flattening of the 1-kHz region. This is demonstrated in Fig. 7.

The obvious way to proceed would be to add another pole pair corresponding to the 7-kHz region. In search for considerably lower Kautz filter orders, compared to the Laguerre equalizer, we however utilize the BU method directly. It provides stable and reasonable pole sets for orders at least up to 40. In Fig. 6 we have presented Kautz equalizers and equalization results for orders 9, 15, 30, and 38. These straightforward Kautz filter constructions are already comparable to the FIR and Laguerre counterparts, but we may further lower the filter orders by omitting some of the poles. For example, for orders above 15, the BU method produces poles really close to \( z = 1 \) because of the low-frequency boost in the target, and omitting some of these poles actually tranquilizes the low-frequency region. We obtain, for example, quite similar equalization results for orders 28 and 34 from the sets with 30 and 38 original poles, respectively (Fig. 8).

To improve the modeling at 1 kHz, we added three to four manually tuned pole pairs to the BU pole sets, corre-

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For a relatively high-quality loudspeaker the deviations in group-delay response are often within 2–3 ms, except for the lowest frequencies, which means that phase equalization is hardly needed.

Kautz filters are inherently well suited for overall equalization of magnitude and phase, for example, by applying LS optimization as discussed.

In this study we have presented the equalization case as an illustrative example of the controllability of frequency resolution rather than the practicality of results. It may not even be desirable to flatten sharp resonances in the main-axis free-field response of a loudspeaker, since off-axis responses can become worse and degrade the overall quality of sound reproduction.
sponding to the resonances in the problematic area. This is actually not too hard since the 1-kHz region is isolated from the dominant pole region, which allows for undisturbed tuning. As starting points we used the 15th- and 30th-order Kautz equalizers of Fig. 6, omitting three pole pairs in the latter case. Three pole pairs were tuned directly to the three prominent resonances, and one pole pair was assigned to improve the modeling below the 1-kHz region. Results for final filter orders 23, 32, and 34 are displayed in Fig. 9, where the last two differ only in the optional compensating pole pair.

Finally we abandon the pole sets proposed by the BU method and try to tune 10 pole pairs manually to the target response resonances. The design is based on 10 selected resonances, represented with 10 distinct pole pairs, chosen and tuned to fit the magnitude response. This is of course somewhat arbitrary, but it seems to work. In Fig. 10, along with the equalizer and target responses, there are vertical lines indicating pole-pair positions. This is clearly one form of “parametric equalization” with second-order blocks since each resonance is represented with a single pole pair. However, with Kautz filters we have, at least to some extent, separated the choice of the resonance structure and the fine-tuning produced by the

Fig. 7. Comparison of 100th-order Laguerre and Kautz equalization results. Kautz filters with 50 complex-conjugate pole pairs correspond to 1-kHz pole angles, and pole radius varied from 0.5 to 0.9 in steps of 0.05.

Fig. 8. Kautz equalization results for orders 28 and 34, with pruned BU poles. Compare with orders 30 and 38 in Fig. 6.

Fig. 9. Kautz equalizers and equalization results for orders 23, 32, and 34, with combinations of manually tuned and BU-generated poles.

Fig. 11 compares some of the Kautz equalization results to those achieved with FIR and Laguerre equalizers of orders 200 and 100, respectively. If we are only concerned with the number of arithmetic operations at run time, we should compare Laguerre and Kautz equalizers at about the same filter orders. The actual complexity depends on many details, but in any case we have achieved very low-order and accurate Kautz equalizers. Furthermore, the low filter orders enable in principle filter transformations to other structures, possibly more efficient ones, such as direct-form filters or otherwise preferable structures.

3.2 Case 2: Room Response Modeling

Models for a room response, that is, a transfer function or an impulse response from a sound source to an observation location in a room, are used for different purposes in audio signal processing, typically as part of a larger system. Room response modeling may constitute a major computational burden, because of both the complexity of target responses and the difficulty of incorporating proper perceptual criteria in those models.

An obvious difficulty in modeling a room response is that the duration of the target response is usually long and that the time-frequency structure is very complicated. Physically speaking there are low-frequency modes determined essentially by the room dimensions and, on the other hand, a reverberation structure produced by the multitude of reflections. There are methods proposed to take into account various time- and frequency-domain modeling aspects as LTI digital filter models [51], [52], including also reverberation designs that approximate reflections and reverberation by complicated parallel and feedback structures [53]. Thus it is interesting to see how a Kautz filter, a generalized transversal filter, can perform similar tasks.

We have chosen as an example a measured room impulse response (re)sampled at 22 050 Hz. The target response for modeling, shown in Fig. 12, is composed of the original signal by omitting the early delay and by truncation to 8192 samples, which yields a duration of 372 ms. Such a target is clearly not an ideal Kautz modeling task since the early response is not a pure superposition of coincident damped exponentials. However, if the Kautz filter is long enough, it should in theory be able to model all the temporal details, though with a potential inefficiency, for example, in producing pure delays. After all, functions in Eq. (1) define an exact representation of any (finite-energy) signal, so this is a brute FIR-type design, but with a more delicate choice of basis functions.

It turns out that the proposed iterative pole optimization method, the BU algorithm, is in trouble in accurate modeling of the full audio frequency range and temporal decay in the form of a single Kautz filter. Therefore we first focus on the low-frequency range where each prominent mode may have a noticeable perceptual effect.

11 The room has approximate dimensions of $5.5 \times 6.5 \times 2.7$ m$^3$ and shows relatively strong modes with long decay times at low frequencies.
Fig. 13 plots the magnitude spectrum of the room response up to 220 Hz at the top. Below it a pole angle bifurcation plot is shown as a function of the Kautz filter order, produced by the BU method, applied to the downsampled target response of 512-sample duration, and for model orders 1–120.

Fig. 14 illustrates the accuracy of the magnitude response compared to the target response (top curve) for Kautz model orders 20–100 in steps of 5. For orders 100–120 the match is practically perfect.

The quality of time–frequency modeling can be checked by a waterfall plot (cumulative spectral decay), as shown in Fig. 15 for the target response and for a Kautz model of order 80. In careful comparison it can be noticed that less prominent modal resonances are weaker and shorter in the model response. Increasing the filter order to 120 makes the model practically perfect in a time–frequency plot too.

As mentioned, full-bandwidth Kautz modeling is not possible by directly utilizing the BU method because of numerical limitations in the iterative algorithm and in root solving of the poles for filter orders above 200–300. If the modeling accuracy can be compromised at some frequencies, or only a spectral envelope model is needed, several Kautz pole determination techniques can be applied. Fig. 16 illustrates magnitude responses of a 320-order model (top curve) and the target response (bottom curve) when Kautz poles are positioned logarithmically in the frequency and with a constant pole radius of 0.98. At low frequencies below 200–300 Hz the magnitude response fit is quite complete, but the poor performance in modeling the dense resonance structure at higher frequencies is evident.

Note that by using a large set of weak poles, the Kautz filter acts like a “slightly recursive FIR filter,” and there is a clear transition from this kind of FIR-type fit of the early response to representing resonances by corresponding pole pairs. In responses where the high-frequency components are short in time we could actually force some of the poles to $z_i = 0$, that is, use a mixture of Kautz and FIR filter blocks. It is noteworthy that this may be done in any permutations and that the same chosen filter parameterization applies also to the FIR blocks.

The application of the BU method to the room response is studied in Fig. 17. A 318th-order Kautz model is presented in Fig. 17(a). It can be seen that the modeling resolution is worse than the target resolution at all frequencies. By utilizing an intermediate warping step in the design phase, better modeling of the lower frequencies is achieved. In Fig. 17(b) Kautz model poles of the 240th

12 In the examples that follow we have used the sampling rate of 22 050 Hz and a bandwidth of approximately 10 kHz.
13 The method was outlined in Section 2.4. The target signal is warped prior to the pole optimization, and then the produced poles are mapped back to the original frequency domain according to the inverse all-pass mapping.
Fig. 15. Cumulative spectral decay plot in frequency range 0–220 Hz. (a) Target. (b) Order 80 modeled room response.

Fig. 16. Magnitude responses for room. 320th-order Kautz model (top) and target response (bottom) plotted with vertical offset. Poles are placed with logarithmically spaced angles and constant radius 0.98. Pole-pair positions are indicated by vertical lines.

Fig. 17. Kautz model magnitude responses produced using BU method with upward offset to target response. (a) Filter order 318. (b) Warping and back-mapping at filter order 240.
order are produced using the appropriate Bark-warping parameter value $\lambda = 0.646$ [26]. As a tradeoff, poorer modeling of the higher frequencies is evident.

It should be noted that this was not a detailed case study on room response modeling and that especially the temporal evolution of the full-band model should be investigated more carefully. In principle it is possible to apply different kinds of partitioning, such as modulation and decimation techniques, to get a partial set of poles, and then decompose a full model. One such approach is through subband modeling, for example, in critical bands [54]. In addition Kautz models for the lower part of the frequency scale can be supplemented with different approaches for the high-frequency modeling.

### 3.3 Case 3: Guitar Body Modeling

As another example of high-order distributed-pole Kautz modeling we approximate a measured acoustic guitar body response sampled at 22 kHz (Fig. 18). The impulse response was obtained by tapping the bridge of an acoustic guitar with an impulse hammer [55], [56], with strings damped.$^{14}$ A guitar body response is typically like a room response of a small room, but since the density of the modes is proportional to the volume, the body response is actually a better justified Kautz modeling task than the room response modeling case discussed in Section 3.2.

The obvious disadvantage of a straightforward FIR filter implementation for the body response is that modeling the slowly decaying lowest resonances requires a very high filter order (1000–3000). All-pole and pole-zero modeling are the traditional choices to improve the flexibility of spectral representation. However, model orders remain problematically high, and the basic design methods seem to work poorly. A significantly better approach is to use separate IIR modeling for the slowly decaying lowest resonances combined with the FIR modeling [57]. Perceptually motivated warped counterparts of all-pole and pole-zero modeling pay off, even in technical terms [55], but in this study we want to focus on the modeling resolution more freely.

Direct all-pole and pole-zero modeling were found to produce unsatisfactory pole sets $\{z_i\}$ for the Kautz filter, even in searching for a low-order substructure. On the other hand, it is relatively easy to find good pole sets by direct selection of prominent resonances and proper pole radius tuning. We will demonstrate next some Kautz modeling cases, with unwarped and warped pole determination based on the BU method. In all cases the poles have been determined from a minimum-phase version of a given body response. Also the model weight coefficients have been obtained by fitting to the minimum-phase version. If the perceptually best body response model is desired, it is better to apply the final fitting to the original version (but with the initial delay removed). This may require a slight increase in filter order, although Kautz filters are inherently well suited to this due to good model fitting in the time domain.

Fig. 19 demonstrates that the proposed pole position optimization scheme, the BU method, is able to capture essentially the whole resonance structure. The Kautz filter order is 262, and the poles are obtained from a 300th-order BU pole set, omitting some poles close to $z = -1$. In general, the BU method works quite well, at least up to an order of 300, and the lower limit for finding the chosen prominent resonances is about 100.

The relatively high order of the model shown in Fig. 19 is required to obtain a good match of the lowest (strongest) modal resonances. The direct application of the BU method pays on average too much attention to the high frequencies compared to the importance of the lowest modes, both physically and from a perceptual point of view. A better overall balance of frequency resolution can be achieved with a lower order by applying the BU method of the Kautz pole determination to a frequency-warped version of the measured response. Fig. 20 depicts the magnitude response of a 120th-order Kautz model obtained this way. A warping coefficient value of $\lambda = 0.66$ is applied, corresponding approximately to the Bark scale warping [26].

Fig. 21 gives further information about the modeling power of the warped BU method by depicting magnitude responses for a set of low-order models (order = 10, 18, 39, 60, and 90), and comparing with the original response. As with any reverberant system, the magnitude response does not tell the full story of how we perceive the response. Comparing time–frequency plots may be needed to evaluate both the temporal and the spectral evolution of modeled responses.

### 4 DISCUSSION AND CONCLUSIONS

Kautz filter design can be seen as a particular IIR filter design technique. The motivation from an audio-processing

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$^{14}$A more “natural” impulse response would be achieved by extracting it by deconvolution from an identification setup using spectrally rich real playing of the acoustic guitar as excitation [57], although the signal-to-noise ratio is better through an impact hammer measurement.
perspective is twofold. Many audio-related target responses can be modeled well by a combination of distributed decaying exponential components, which by definition is what a Kautz filter does, but in an orthonormalized form, providing many favorable properties. On the other hand, Kautz filtering techniques provide many ways to incorporate an auditorily meaningful allocation of frequency resolution to the modeling.

The cases of modeling and equalization were selected as audio examples to show the applicability of Kautz filters. Many specific questions, such as the audio engineering relevance of the modeling details, perceptual aspects of the designs, as well as computational robustness and expense have been addressed only briefly or not at all. These details call for further investigations.

The aim of this study was to show that it is possible to achieve good modeling or equalization results with lower Kautz filter orders than, for example, with warped (Laguerre) or traditional FIR and IIR filters. In the loudspeaker equalization case Kautz filter orders of 20–30 can achieve similar results of flatness as warped IIR equalizers of order 100–200, or much higher orders with FIR equalizers. This reduction is due to a well-controlled focusing of frequency resolution on both the global shape and in particular on local resonant behaviors.

The room response modeling case should be taken as a mere methodological study, except possibly regarding the usefulness of obtained low-frequency models. In the case of guitar body response modeling the low-frequency modes are important perceptually, and low-order Kautz filters can focus sharply on them, showing an advantage over warped, IIR, and FIR designs, especially when the focus is on separate low-frequency modes of the body response.

In this study we have successfully applied the BU method of Kautz pole determination, with and without frequency warping. There are numerous other possible techniques and strategies to search for an optimal model for a given problem, including exhaustive search for very low-order models and, for example, genetic algorithms for somewhat higher orders. Different tasks can be solved best with different approaches. The cases investigated here just hint on general guidelines, and a fully automatic search for optimal solutions, even in the cases presented, requires further work. We, however, have demonstrated the potential applicability of Kautz filters. They are found to be flexible generalizations of FIR and Laguerre filters, providing IIR-like spectral modeling capabilities with well-known favorable properties resulting from the orthonormality. The competitiveness compared to Laguerre modeling is based on the fact that the generalization step imposes little or no extra computation load at runtime, even if the design phase may become more complicated.

MATLAB scripts and demos related to Kautz filter design can be found at http://www.acoustics.hut.fi/software/kautz.

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6 REFERENCES


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