FINITE DIFFERENCE METHOD VS. DIGITAL WAVEGUIDE METHOD IN STRING INSTRUMENT MODELING AND SYNTHESIS

Cumhur Erkut and Matti Karjalainen

Helsinki University of Technology
Laboratory of Acoustics and Audio Signal Processing
P.O. Box 3000, Fin-02015 Espoo, Finland
`Cumhur.Erkut@hut.fi`, `Matti.Karjalainen@hut.fi`

Abstract
The one-dimensional digital waveguides, combined with the commuted synthesis method, allow modeling and high-quality synthesis of plucked string instrument tones in a very efficient manner. However, the increasing computational power of the modern processors makes it feasible to experiment with more complex algorithms also for real-time sound synthesis purposes. By certain simplifications, time-domain methods based on finite differences (FDTD) are efficient enough to run in real time on modern processors and yet they are more flexible than the computationally less expensive commuted synthesis. The resulting structures, which are called 1-D FDTD waveguides, have previously been shown to be equal to or to approximate many properties of digital waveguides, including lossless and lossy propagation, input and output ports, terminations, and scattering junctions. The numerical stability of the 1-D FDTD waveguides, as well as their initialization and formation of the traveling waves are well understood. However, a careful comparison between 1-D FDTD waveguides and conventional digital waveguides in terms of their limitations, computational efficiency, accuracy, and interaction has not been carried out. The aim of this paper is to fill this gap by highlighting important properties of both methods side by side in string instrument modeling and synthesis. We then try to combine the best properties of both methods and discuss the interaction of the two model structures. Synthetic tones and short musical phrases obtained by both synthesis models will be demonstrated during the presentation. These sound examples will also be available at `http://www.acoustics.hut.fi/demos/`.

INTRODUCTION

Model-based computational algorithms that mimic the sound production in musical instruments are progressively being an important tool in musical acoustics. In the case of finite difference models, where the governing partial differential equations are converted to difference equations and numerically integrated, a full-scale guitar model that simulates the vibrations of the strings and the body, as well as the radiation has been recently reported [1]. The model simulates the dynamic interactions during plucking and other important temporal mechanisms with a good accuracy. However, the computational load of the method prevents its use for real-time sound synthesis.

An efficient technique for model-based sound synthesis of musical instruments in general, and of plucked string instruments in particular, is the digital waveguide technique [2, 3]. Combined with the commuted synthesis method [4, 5], where the dynamic interaction between finger and the string, as well as the body response are merged into a wavetable, a full scale commuted waveguide synthesis model of the classical guitar [6] and other plucked strings instruments have been reported [7].

By certain simplifications, time-domain methods based on finite differences (FDTD) can be made efficient enough to run in real time on modern processors. The resulting DSP structures, which are called 1-D FDTD waveguides [8], are dynamically more flexible than the computationally less expensive commuted synthesis models.

1D-FDTD string models have previously been shown to be equal to or to approximate many properties of digital waveguides,[1] including lossless and lossy propagation, input and output ports, terminations, and scattering junctions [8]. The physical and mathematical properties of a 1D-FDTD waveguide are further investigated in a follow-up paper [9], where a sufficient condition for the numerical stability of the model has been derived and the relationship between the 1-D FDTD waveguide and the lossy wave equation has been demonstrated. In addition, traveling *slope* waves have been obtained from the past states of the 1D-FDTD model.

This paper provides a comparison between the 1-D FDTD waveguides and DWGs and focuses on the formulation of their interaction. After presenting basic theory of both methods, a comparison in terms of their limitations, computational efficiency, and accuracy has been carried out. We then combine the best properties of both methods and discuss the interaction of the two model structures. The structure of the paper follows these guidelines. Sound examples that demonstrate the resulting hybrid model are available at `http://www.acoustics.hut.fi/demos/`.

BACKGROUND

In this section, we briefly review the basic theory of the 1-D FDTD waveguides and DWGs. The reader is referred to references [8, 9] for a more detailed discussion of the former method, and to reference [3] for an in-depth coverage of the latter.

Basic Theory of 1-D FDTD Waveguides

The formulation of the 1-D FDTD waveguides originates from the finite difference approximation of the wave equation in a one-dimensional lossless medium [10]:

$$y_{tt} = c^2 y_{xx} \tag{1}$$

where y is the displacement, t and x are temporal and spatial variables, respectively. The propagation speed is given by $c = \sqrt{T/\mu}$, where T is the tension of the string and μ is its linear mass density. The following central difference schemes for even order partial derivatives can be used for discretization of Eq. (1) [11]:

$$y_{tt} \approx \frac{y_{x,t+\Delta t} - 2y_{x,t} + y_{x,t-\Delta t}}{\Delta t^2} \tag{2}$$

$$y_{xx} \approx \frac{y_{x+\Delta x,t} - 2y_{x,t} + y_{x-\Delta x,t}}{\Delta x^2} \tag{3}$$

In the discrete form, the temporal and spatial variables may be denoted as $k = x/\Delta x$ and $n = t/\Delta t$, where Δt and Δx are the temporal and spatial sampling intervals, respectively. The sampling intervals cannot be chosen arbitrarily; the Von Neumann stability condition [11, 12] dictates the following constraint

$$c\frac{\Delta t}{\Delta x} \leq 1. \tag{4}$$

The choice $c = \Delta x/\Delta t$ eliminates the truncation error and numerical dispersion [11]. In this case, the term $y_{k,n}$ vanishes and the discrete solution of Eq. (1) yields

$$y_{k,n+1} = y_{k-1,n} + y_{k+1,n} - y_{k,n-1} \tag{5}$$

Note that this choice is valid in the case of an ideal string. If the medium is non-ideal, spatial oversampling may improve the simulation accuracy.

[1] We will refer to a digital waveguide presented in [2] as **DWG** henceforth to avoid confusion. The DWGs are based on the traveling wave solution of the wave equation, whereas 1-D FDTD waveguides do not presume the form of the solution. An excellent review of the DWG theory can be found in [3]

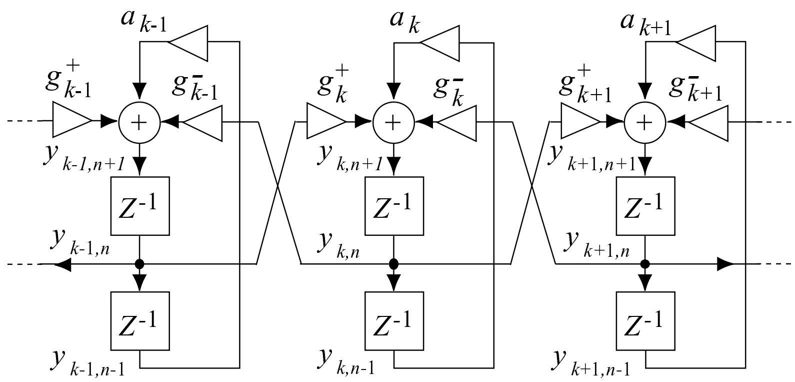

Figure 1: *Digital waveguide structure based on finite difference approximation of one-dimensional wave equation.*

The 1-D FDTD waveguide is a generalization of the solution in Eq. (5). It has been defined as the following DSP structure [8]

$$y_{k,n+1} = g_k^+ y_{k-1,n} + g_k^- y_{k+1,n} + a_k y_{k,n-1}. \tag{6}$$

This DSP block is illustrated in Fig. 1.

In [8], it has been stated that the parameters g_k^+, g_k^-, and a_k can be used for simulating the losses, scattering and fractionally positioned terminations. If $g_k^+ = g_k^- = g$, and $a_k = a$, the following condition is sufficient for the stability [9]

$$a = -g^2 \quad , \quad |g| \leq 1. \tag{7}$$

It has been shown that this selection corresponds to the numerical integration of the lossy wave equation, where the losses are modeled by frequency-independent loss terms [9]. In the following section, we will derive the same equivalence using a different formulation based on the DWG theory. For this purpose, we present a short review of the DWG theory below. For a in-depth theory of the DWGs, the reader is referred to [3].

Basic Theory of DWGs

The DWG theory is based on the traveling-wave solution of the wave equation. With the selection $c = \Delta x/\Delta t$, the traveling wave components obeying the ideal lossless wave equation in Eq. (1) can be simulated by pure delays, and the total displacement is computed by adding the traveling wave components.

In any physical string there are energy losses due to fluid and viscous damping, and yielding terminations. The losses may be approximated by a small number of odd-order time derivatives added to the ideal wave equation [3, 11]. In the simplest case, a first-order temporal derivative accounts for frequency-independent losses, and its coefficient b_1 controls the decay rate of the traveling wave components. Thus, the following modified traveling wave solution can be shown [3] to satisfy the lossy wave equation (see for instance [9])

$$y(x,t) = e^{-\frac{-b_1/2\mu}{t}} \left\{ y_r(x-ct) + y_l(x+ct) \right\} \tag{8}$$

where μ is the linear mass density of the string. Sampling these exponentially decaying traveling waves gives

$$y(n,k) = g^{-k} y^+(n,k) + g^{-k} y^-(n,k), \tag{9}$$

where $g = e^{-b_1 T/2\mu}$. The corresponding DWG structure is shown in Fig. 2.

Vast computational savings may be realized by *commuting* all the arithmetic operations out of unobserved and undriven sections of the medium and *consolidating* them at minimum number of points. Thus, frequency-dependent losses, dispersion, and fractional delays may be

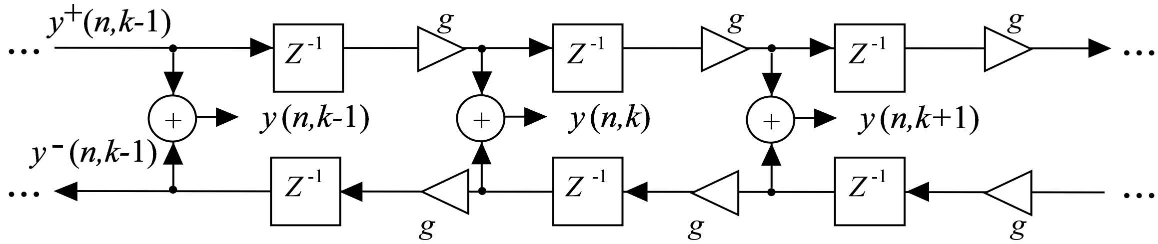

Figure 2: *Lossy digital waveguide. After Smith* [3].

realized by using low-order digital filters, connected prior to the input or output junctions [3]. The drawback, however, is the deviation of the signals *inside* the DWG structure from their exact physical values. This corresponds to a reduction of the spatial accuracy inside the DWG. Such a reduction of the spatial accuracy may be important, e.g., in the presence of distributed excitation mechanisms or distributed nonlinearities.

DWG VS. 1-D FDTD WAVEGUIDES

As stated in [13], the DWG structures can be conceptualized as a refinement of finite differences for efficiency and better numerical accuracy. In this section, we compare the accuracy, computational efficiency, and limitations of the two models.

Accuracy

It is a common practice to prove the equivalence between a DWG and an 1-D FDTD waveguide in the ideal lossless case. The basic idea is to convert the traveling wave variables using the natural operations of the DWG (delays) to the state variables of the finite differences [3, 13], or vice versa [9]. The equivalence of the two methods in the lossy case is a straightforward extension of the ideal lossless case. However the following procedure is illustrative since it clarifies the fundamental steps of the formulation that will follow.

Consider the homogeneous ($g_k = g$ for all k) waveguide section in Fig. 2. Although typically the outputs of a DWG are not computed at every junction, however, for sake of discussion, we will assume that they are available at any instant. Given the traveling waves at the nth instant, at $n+1$ the displacement yields

$$y(n+1,k) = y^+(n+1,k) + y^-(n+1,k) = gy^+(n,k-1) + gy^-(n,k+1) \tag{10}$$

Replacing each traveling wave with the difference of the total wave and the delayed traveling wave of the opposite direction, Eq. (10) becomes

$$\begin{aligned} y(n+1,k) &= g\left\{y(n,k+1) - gy^+(n-1,k)\right\} + g\left\{y(n,k-1) - gy^-(n-1,k)\right\} \\ &= gy(n,k+1) + gy(n,k+1) - g^2y(n-1,k) \end{aligned} \tag{11}$$

We note that in the homogeneous case, Eq.(11) is equivalent to Eq.(6) with $a = -g^2$, which is the sufficient condition for the stability given in Eq. (7).

We thus state that both the DWG and the 1-D FDTD waveguides are equivalent for the homogenous wave equation with a frequency independent damping term. Since the DWG method provides the *exact* solution of the lossy wave equation, we can state that the 1-D FDTD has the same physical accuracy as the DWG method. In the case of finite-precision arithmetic, however, the numerical accuracy of the methods may differ.

In order to understand this, we may concentrate on the last term (local memory) in Eq. (11). Following the steps from Eq. (10) to Eq. (11), it is clear that the local memory ensures that there is no energy cumulation in the kth junction of the 1-D FDTD waveguide. Non-energetic junctions that are inherent in the DWG models are thus achieved by an arithmetic operation in 1-D FDTD models. In other words, 1-D FDTD method is more prone to numerical errors that may be caused by finite-precision arithmetic. However, in our floating-point implementation we did not encounter a numerical problem.

Efficiency

As stated before, the DWGs can be made extremely efficient by consolidation of arithmetic operations (losses, dispersion, etc.) and by leaving out large blocks of pure delays. This implies a trade-off between spatial accuracy and computational efficiency.

The conventional finite difference methods introduce more computations per output sample in favor of spatial accuracy even in the simplest case of frequency independent losses. In more detailed simulations, the computations and memory needs may increase significantly compared to that of the DWG methods. If the dispersion and frequency-dependent losses are included, a single output depends on three past states of the whole structure and on nine spatio-temporal cell values, and is computed by nine cumulative multiply-adds [11].

The main motivation in formulation of 1-D FDTD waveguides is to achieve a better trade-off between spatial accuracy and computational efficiency. As can be checked from Eq. (6), an output is calculated by three cumulative multiply-adds and is dependent only on three spatio-temporal cell values [9].

Limitations

However, the efficiency of the 1-D FDTD waveguide is achieved by limiting its accuracy compared to a more general finite difference model, such as the string model given in [11]. In fact, Eq. (11) reveals that a 1-D FDTD waveguide can only simulate frequency-independent losses. However it can easily handle dynamic distributed interaction, such as an accurate initial state formation [9] or nonlinearity [8]. Note that, although these operations may also be carried out by a DWG, the efficiency of the DWG reduces in such cases.

In order to achieve an optimal trade-off between spatial accuracy and efficiency, a hybrid scheme has been proposed in [9]. In this proposal, as a first step, the traveling slope waves were formulated by using the state variables of the 1-D FDTD model. The traveling waves can then be inserted to a DWG structure, where the frequency-dependent losses, fractional delays, and the dispersion are easier to model. Such a hybrid tries to combine the best properties of both methods. A more direct formulation of the interaction is presented in the next section.

INTERFACING 1-D FDTD AND DWG MODELS

This section formulates an interface that may be used to connect a 1-D FDTD structure to a DWG. Suppose that the string is discretized so that the section $0 < k \leq K$ corresponds to a 1-D FDTD waveguide and $k > K$ corresponds to a DWG. Assume that $g_k^- = g_k^+ = g$ for all k, where g is the loss term given by $g = e^{-b_1 T/2\mu}$ (see Eq. (9)). Using similar operations as in Eq. (10) and Eq. (11) the right-going wave at the interface yields

$$y^+(n+1, K+1) = gy^+(n, K) = gy(n, K) + ay^-(n-1, K+1). \tag{12}$$

The physical displacement wave on the 1-D FDTD side is related to the traveling waves inside the DWG section as follows

$$y(n+1, k) = gy(n, k-1) + g\left\{y^+(n, k+1) + y^-(n, k+1)\right\} + ay(n-1, k) \tag{13}$$

Eqs. 12 and 13 directly imply the structure of a hybrid model shown in Fig. 3. An important goal in object-based sound synthesis is to interconnect individually discretized objects by using special interconnection elements (*adaptors*) [14]. The hybrid model of Fig. 3 may as well be represented as two separate objects and a dynamical interconnection element. This representation is illustrated in Fig. 4.

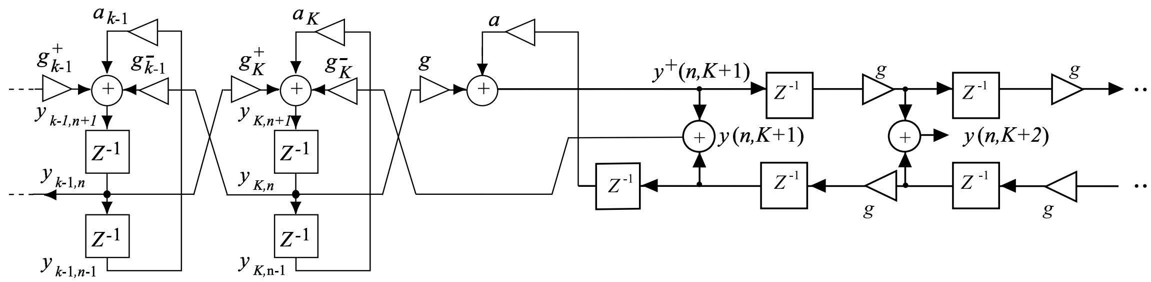

Figure 3: *The interconnection of a 1-D FDTD waveguide to a DWG.*

In order to demonstrate the hybrid block shown in Fig. 3, we conducted the following experiment. We first constructed a 1-D FDTD model of a string with $f_0 = 220.5$ Hz. This corresponds to the spatial length of $k = 100$ when $f_s = 44100$. We then replaced the right half of the model with a DWG structure so that the adaptor is placed at $K = 50$. For sake of simplicity, we imposed rigid boundary conditions (i.e., $y(0,n) = y(99,n) = 0$ for all n) and considered the homogenous lossless case (i.e., $g_k = g = 1$ for all k). Note that lossy propagation as well as other boundary conditions can easily be obtained with the hybrid model as well. The model is initiated by a triangular initial displacement pulse and zero initial velocity. Figure 5 presents the simulation results.

As expected, the initial pulse propagates to two opposite directions. At the instant $n = 15$, the right-going and left-going waves are approaching towards the adaptor and the left boundary, respectively. At $n = 30$ the right-going wave is partly in the DWG section; no scattering is observed as expected. At this instant, the other component is reflected at the boundary, and propagates also towards the adaptor on the right-hand side. At $n = 75$ the first component in the DWG is being reflected, and the second component enters into the DWG. At $n = 135$ the first component leaves the DWG and the second component propagates towards the adaptor. As n increases, the two components propagate continuously in the hybrid structure, which verifies the operation of the adaptor.

Since the adaptor requires only the local traveling waves from the DWG, all operations may be commuted and consolidated within this part. For instance, frequency-dependent losses can be modeled by a lumped digital filter. Distributed interaction, on the other hand, can be simulated on the 1-D FDTD side with ease. The proposed model thus combines the best properties of the two methods.

CONCLUSIONS AND FUTURE PLANS

This paper provided a comparison between 1-D FDTD waveguides and DWGs in terms of their limitations, computational efficiency, and accuracy. The important properties of both methods are indicated. The best properties of both methods are combined in a hybrid model by the interaction of the two model structures. The sound examples available at `http://www.acoustics.hut.fi/demos/` demonstrate the sound synthesis capabilities of the proposed hybrid structure.

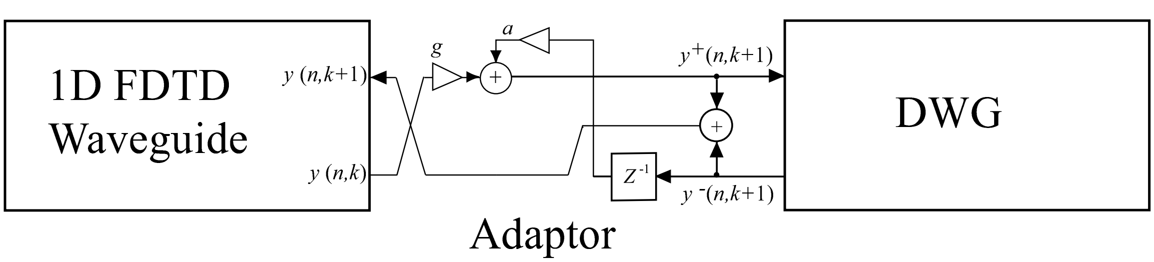

Figure 4: *The adaptor formulation of the corresponding hybrid model.*

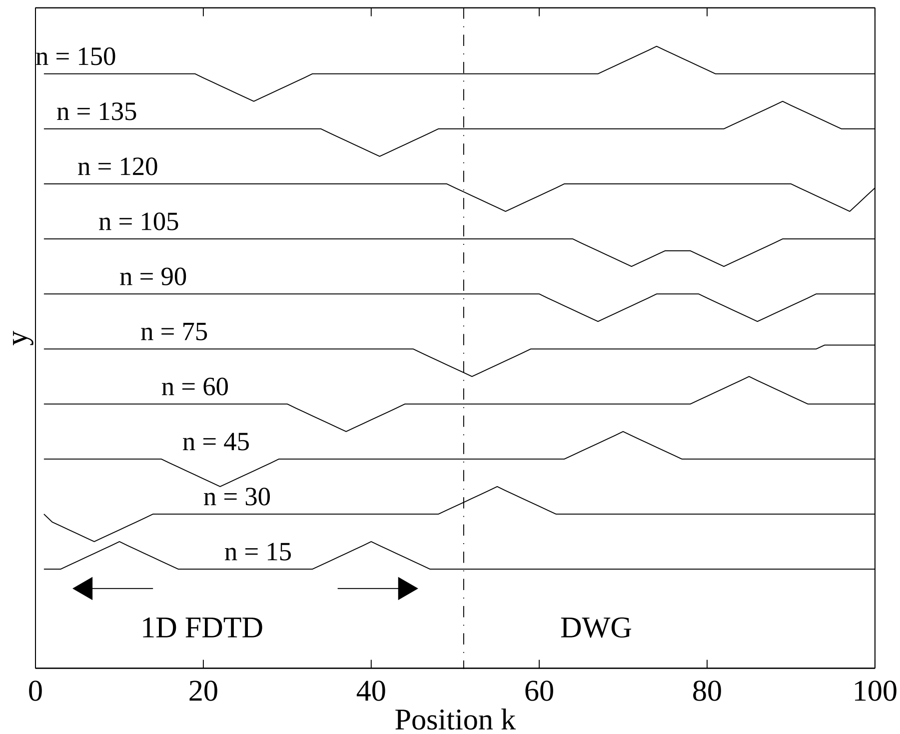

Figure 5: *The displacement of the string at various time instants n as calculated by the hybrid model shown in Fig. 3. The terminations are rigid at both ends. The adaptor is placed at $k = 51$. At the instant $n = 0$ (not shown in the figure) a triangular initial displacement pulse is inserted into the 1-D FDTD side. As n increases, the traveling components propagate in the hybrid structure.*

Although simple and straightforward, the strategy to convert a traveling wave model, such as a DWG model, to a total-variable model (or a K-model [14]), such a 1-D FDTD model may be generalized to other structures. In particular, if the constant g terms are replaced with FIR filters in Eq. (11) on the DWG side, the resulting updated finite difference model can account also for the frequency-dependent damping. The updated model is more accurate than 1-D FDTD, but it requires more memory. If the FIR filters are replaced in turn by IIR filters, the resulting finite difference scheme may be implicit [11]. In this case, a general delay-free loop elimination scheme may be used to make resulting structure explicit [15]. This way of deriving finite difference schemes is more DSP-oriented and can be further investigated in the future. In addition, a general formulation for interconnection of the K-models and the wave models should be elaborated.

Another important future direction is to achieve a scattering formulation for the 1-D FDTD waveguides. The scattering formulation is the backbone of the DWG theory [3] and it provides important advantages when interconnecting multiple sound synthesis objects. A similar formulation for the 1-D FDTD waveguides is an important future goal.

ACKNOWLEDGMENTS

The authors wish to thank Lauri Savioja for fruitful discussions. This work is supported by the IST/ALMA project (ALgorithms for the Modelling of Acoustic interactions, IST-2000-30072).

REFERENCES

[1] Chaigne, A., "Numerical simulations of stringed instruments – today's situation and trends for the future," *Catgut Acoustical Society Journal*, **4**, pp. 12–20, 2002.

[2] Smith, J. O., "Physical modeling using digital waveguides," *Computer Music J.*, **16**, pp. 74–91, 1992.

[3] Smith, J. O., "Principles of digital waveguide models of musical instruments," in *Applications of Digital Signal Processing to Audio and Acoustics* (Kahrs, M. and Brandenburg, K., eds.), pp. 417–466, Boston, Massachusetts, USA: Kluwer Academic Publishers, 1998.

[4] Smith, J. O., "Efficient synthesis of stringed musical instruments," Proc. Int. Computer Music Conf., (Tokyo, Japan), pp. 64–71, 1993.

[5] Karjalainen, M., Välimäki, V., and Jánosy, Z., "Towards high-quality sound synthesis of the guitar and string instruments," Proc. Int. Computer Music Conf., (Tokyo, Japan), pp. 56–63, 1993.

[6] Laurson, M., Erkut, C., Välimäki, V., and Kuuskankare, M., "Methods for modeling realistic playing in acoustic guitar synthesis," *Computer Music J.*, **25**, 2001.

[7] Erkut, C., *Aspects in analysis and model-based sound synthesis of plucked string instruments*. PhD thesis, Helsinki University of Technology, Espoo, Finland, 2002. Available online at `http://lib.hut.fi/Diss/`.

[8] Karjalainen, M., "1-D digital waveguide modeling for improved sound synthesis," Proc. IEEE Int. Conf. Acoustics, Speech and Signal Proc., vol. 2, (Orlando, Florida, USA), pp. 1869–1872, 2002.

[9] Erkut, C. and Karjalainen, M., "Virtual strings based on a 1-D FDTD waveguide model," Proc. AES 22nd International Conference, (Espoo, Finland), pp. 317–323, 2002.

[10] Fletcher, N. H. and Rossing, T. D., *The Physics of Musical Instruments*. New York, USA: Springer-Verlag, 2nd ed., 1998.

[11] Chaigne, A., "On the use of finite differences for musical synthesis. Application to plucked stringed instruments," *Journal d'Acoustique*, **5**, pp. 181–211, 1992.

[12] Strikwerda, J. C., *Finite difference schemes and partial differential equations*. California, US: Wadsworth, Brooks & Cole, 1989.

[13] Bilbao, S. D., *Wave and scattering methods for the numerical integration of partial differential equations*. PhD thesis, Stanford University, California, USA, 2001.

[14] Pedersini, F., Sarti, A., and Tubaro, S., "Object-based sound synthesis for virtual environment using musical acoustics," *IEEE Signal Processing Magazine*, **17**, pp. 37–51, 2000.

[15] Borin, G. and G. De Poli, D. R., "Elimination of delay-free loops in discrete- time models of nonlinear acoustic systems," *IEEE Transactions on Speech and Audio Processing*, **8**, pp. 597–605, 2000.