Recent Advances in Multi-Paradigm Physical Modeling

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Physics-based modeling of sound sources, such as of musical instruments and speech production, is based on several different modeling paradigms or methodologies. These can be roughly categorized as finite difference schemes, mass-spring models, modal decomposition methods, digital waveguide models, wave digital filter models, and source-filter models. The three first methods are based on the use of so-called K-quantities, while the next two ones use wave quantities for computation, and the last one is a category between true physical modeling and signal processing algorithms. Research groups in physics-based modeling have traditionally applied only their one or two favorite paradigms. Recently there has been growing interest to utilize any or many of the approaches, according to specific needs. This includes also combining the paradigms to hybrid modeling in order to optimize the performance or flexibility of design. The trend has improved the overall understanding of physics-based modeling, and will probably lead to many new models and ideas, as well as to tools that support multi-paradigm approaches. This paper presents an overview of multi-paradigm techniques in physic-based modeling, as applied particularly to musical instruments. It will also discuss some recent work in the Sound Source Modeling Group at the Helsinki University of Technology.

1 Introduction

Real-world systems of interest in music acoustics and computer music are typically continuous in time and space, and therefore their inherently dynamic behavior is described by partial differential equations [1]. Computer-based modeling and simulation of them requires, however, discretization of the underlying PDEs, which in a general case will correspond to the continuous analog systems only when the sampling rate approaches infinity, i.e., temporal and spatial sample intervals are made infinitesimally small.

Discretization in time and space leads to interesting and difficult problems that are not found in the ordinary continuous case. Particularly when systems are *simulated* or *synthetized*, i.e., computed efficiently in the time domain by discrete techniques, in contrast to being *solved* from equations, the question of *localized discretization* (block-wise construction of models through interconnection of elements) and consistent *scheduling* (ordering of operations) are of major importance. It is advantageous to formulate discrete-time computations in terms of *digital signal processing* (DSP) argorithms that are optimized for computational efficiency and robustness [2].

In the analog world we may in the limit assume arbitrarily short (infinitesimal) but non-zero delays between spatial points of interest, and the order of events follows the causality principle of physics. In the discrete-time world, however, a single sample period is the shortest possible non-zero time interval, in which an explicit two-way physical interaction can happen. This leads easily to the problem of *delay-free loops*, i.e., implicit equations where the computation of the output of an operation needs an input value that may be dependent on the output value not yet known. Particularly with nonlinear elements this is a fundamental problem [3]. Another fundamental problem with nonlinearities in discrete-time computation is aliasing of frequency components [2].

The delay-free loop problem is easier to overcome if computations are formulated by wave quantities instead of ordinary physical quantities [4]. Referring to electric circuits, the latter ones are often called the *Kirchhoff quantities*, in contrast to *wave quantities*. Using dual K-variables (K for Kirchhoff), such as voltage and current or force and velocity, is very intuitive for circuits and their mechanical equivalents, but in the discrete methodology they are not as easy to use. K-elements, formulated as "transfer functions" between dual K-variables, cannot form circuits and networks directly, because for example voltage across and current through an electric component don't explicate causality. Both of them depend on a full circuit topology so that only their interrelation is fixed by the port impedance of the element. Causality is obtained if the K-quantities are transformed to W-quantities, i.e., into wave-based formulation, in order to compute them by explicit causal relations through local interactions. This is different from solving system equations which permits global (even delay-free) interactions.

In this paper we discuss several discrete-time modeling paradigms, particularly the *digital waveguides* (DWGs), *wave digital filters* (WDFs), and *finite difference time domain schemes* (FDTDs). Their properties are compared and their mixing to hybrid modeling techniques are probed further from previous studies. Examples characterize the realization principles as applied to musical instruments.

2 Discrete-time paradigms

For spatially distributed systems the wave equation is the most fundamental starting point. In a one-dimensional lossless medium the wave equation is written

$$y_{tt} = c^2 y_{xx} \tag{1}$$

where y is (any) wave variable, subscript tt refers to second partial derivative in time t, xx second partial derivative in place variable x, and c is speed of wavefront in the medium of interest. For example in a vibrating string we are primarily interested in transversal wave motion for which $c = \sqrt{T/\mu}$, where T is tension force and μ is mass per unit length of the string [1]. In the case of acoustic wave propagation, Eq. (1) holds for lossless longitudinal plane waves.

2.1 Digital waveguides

Digital waveguide modeling is based on the fact that wave propagation in a medium can be simply and efficiently simulated by two-directional delay lines [5], one delay-line for each directional wave component. This is closely related to the d'Alembert solution of the wave equation

$$y(t, x) = y^+(t - x/c) + y^-(t + x/c) \tag{2}$$

where $+$ and $-$ denote the right-going and the left-going components of the total waveform. Assuming that the signals are bandlimited to half of the sampling rate, we may sample the traveling waves without losing any information by selecting T as the sample interval and X the position interval between samples so that $T = X/c$. From this it follows that wave propagation can be computed by updating state variables in two delay lines by

$$y^+_{k,n+1} = y^+_{k-1,n} \quad \text{and} \quad y^-_{k,n+1} = y^-_{k+1,n} \tag{3}$$

i.e., by simply shifting the samples to the right and left, respectively. This kind of discrete-time modeling is called *Digital Waveguide* (DWG) modeling [5]. Figure 1 characterizes a simple 1-D digital waveguide which could describe for example a vibrating string.

Low-pass and all-pass filters can be cascaded with delay elements to simulate damping and dispersion. Delay sequences between points of signal observation (output) and feed-in (input) can be consolidated into subsystems that are computationally highly efficient [5]. In *Commuted waveguide synthesis* this is utilized by arranging the elements of a string instrument, i.e., plucking, string, and body, so that plucking and body are realized as a wavetable that feeds a string model.

In addition to delays, junctions connecting elements are needed that fullfil physical continuity constraints, i.e., the Kirchhoff rules. For a parallel junction of acoustic components we may write that $P_i = P_\mathrm{J}$ and $U_\mathrm{ext} + \sum_i U_i = 0$

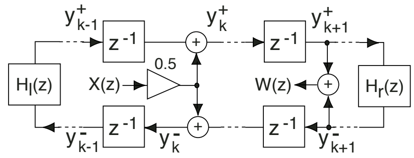

Figure 1: Digital waveguide structure based on traveling wave components from d'Alembert solution of lossless wave equation. $X(z)$ is input excitation such as plucking force of a string and $Y(z)$ is output such as transversal string velocity. Wave reflection at terminations is modeled by reflection filters $H_\mathrm{l}(z)$ and $H_\mathrm{r}(z)$.

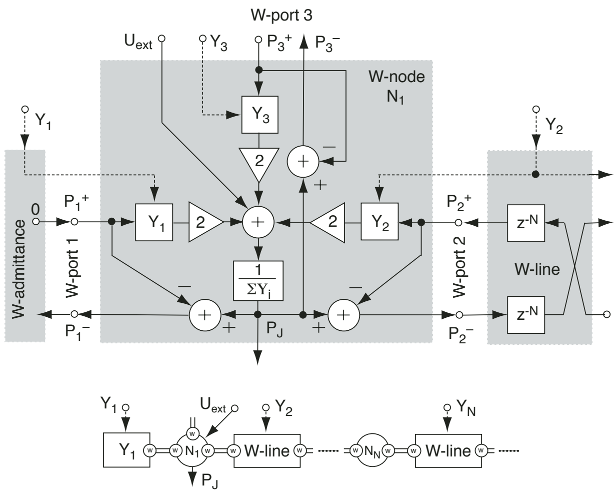

Figure 2: *Top*: A 3-port parallel scattering junction for acoustic pressure waves. Incoming pressures are P_i^+, outgoing ones P_i^+, and P_J is common junction pressure. Port 1 (left) is terminated by admittance Y_1, port 2 (right) is connected to a delay-line having wave admittance Y_2, and port 3 (top) is not connected. *Bottom*: Block diagram with abstracted blocks and how they can be connected to form a 1-D DWG waveguide.

where P_i are pressures and U_i volume velocities at the ports of the junction, P_J is the common pressure of coupled branches and U_ext is an external volume velocity to the junction. When port pressures are represented by incoming wave components P_i^+ and outgoing wave components P_i^-, admittances attached to each port by Y_i, and

$$P_i = P_i^+ + P_i^- \quad \text{and} \quad U_i^+ = Y_i P_i^+ \tag{4}$$

the junction pressure P_J can be obtained as:

$$P_\mathrm{J} = \frac{1}{Y_\mathrm{tot}} (U_\mathrm{ext} + 2 \sum_{i=0}^{N-1} Y_i P_i^+) \tag{5}$$

where $Y_\mathrm{tot} = \sum_{i=0}^{N-1} Y_i$ is the sum of all admittances to the junction. Outgoing (scattered) pressure waves, obtained from Eq. (4), are then $P_i^- = P_\mathrm{J} - P_i^+$. Figure 2 (top) depicts this as a signal flow diagram for the computation of such a scattering junction. The bottom part of

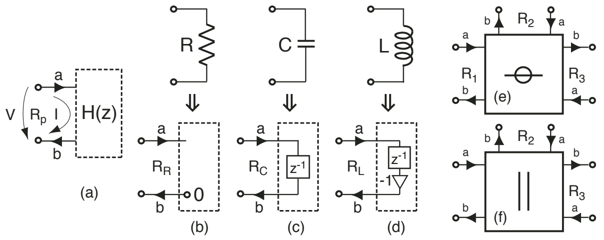

Figure 3: Wave digital filter components: (a) 1-port component generically, (b) WDF resistor $R_{\mathrm{R}} = R$, (c) WDF capacitor $R_{\mathrm{C}} = T_{\mathrm{s}}/2C$, (d) WDF inductor $R_{\mathrm{L}} = 2L/T_{\mathrm{s}}$, (e) 3-port series adaptor, and (f) 3-port parallel adaptor.

Fig. 2 depicts an abstraction diagram of these blocks. Arbitrary admittances can be given by their z-transforms as digital filters (FIR or IIR filters, or just as real coefficients if all impedances are real). Notice that each admittance is also represented in the term $1/\sum Y_i$.

Any number of element ports can be connected to a scattering junction of the type shown in Fig. 2. This makes it possible to form mesh-like structures where two-port delay-lines are used between nodes. Such structures are useful for constructing *digital waveguide meshes* in two and three dimensions [6].

2.2 Wave digital filters

Wave digital filters (WDFs) are models that were originally developed for discrete-time simulation of lumped element circuits and systems, as known from the theory of analog electric circuits [4]. The close relationship between them and digital waveguides is now well known [5, 7]. While DWGs emphasize delays and wave propagation, WDFs have emphasis on lumped element modeling. However, both are capable to both types; they are compatible and complementary approaches for wave-based modeling. The WDF formalism is based on a notation of ('voltage') waves a and b as

$$a = V + R_{\mathrm{p}} I, \qquad b = V - R_{\mathrm{p}} I \tag{6}$$

where a is in-coming and b is out-going wave in a port, V is voltage and I is current as Kirchhoff variables, and R_{p} is port resistance (reference resistance), as denoted in Fig. 3(a). Figures 3(b-f) present the most basic elements used in wave digital filters: resistor, capacitor, inductor, series adaptor, and parallel adaptor. The two latter ones are used to construct circuit models by series and parallel connections of the component blocks. There are several other important circuit elements available for WDFs [4], and the wave digital filter theory has also been extended to multidimensional circuits and systems [7].

The wave digital resistor in Fig. 3(b) is simple, and as for a resistive DWG junction in Fig. 2, no real computation is needed since there is no feedback path from wave input a to output b.

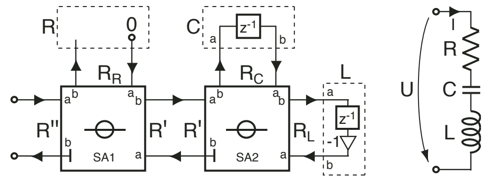

Figure 4: (Left) A WDF series connection of resistor (R), capacitor (C), and inductor (L) constructed by two three-port series adaptors (SA1 and SA2). (Right) Equivalent analog circuit.

Analog capacitors and inductors can only be approximately represented when using discrete-time techniques. The most appropriate way is to use the structures shown in Fig. 3(c) for a WDF capacitor and in Fig. 3(d) for a WDF inductor. These feedbacks through a unit delay correspond to a bilinear mapping of the analog domain (Laplace domain) to discrete-time domain (z-domain). For the WDF capacitor C in Fig. 3(c) we get

$$Z_{\mathrm{C}}(z) = \frac{1+z^{-1}}{1-z^{-1}}\, R_{\mathrm{C}}, \qquad R_{\mathrm{C}} = T_{\mathrm{s}}/2C \tag{7}$$

where R_{C} is port resistance and T_{s} is the sampling period applied. Correspondingly for an inductor L we get

$$Z_{\mathrm{L}}(z) = \frac{1-z^{-1}}{1+z^{-1}}\, R_{\mathrm{L}} \qquad R_{\mathrm{L}} = 2L/T_{\mathrm{s}} \tag{8}$$

In Fig. 4, a WDF model for a series connection of inductor L, capacitor C, and resistor R is depicted. It is constructed by two three-port *series adaptors* (SA1 and SA2) that implement wave scattering according to Kirchhoff laws. The adaptors in the circuit model also show that delay-free loops are avoided by impedance-matched reflection-free ports denoted by $\dashv$ in the adaptors. For more details, see [4].

The WDF models can be used also in other physical domains, such as the mechanical and acoustical ones. In a direct analogy of a mechanical resonator, Fig. 4 can be interpreted as a mass-spring-damper system, where all components have the same velocity, force corresponding to voltage, velocity to current, mass to inductance, spring constant to capacitance, and damping coefficient to resistance. In the acoustical domain, force is replaced by pressure and velocity by volume velocity.

2.3 Finite difference models

Finite difference approximation is a popular method of numerical integration of partial differential equations [8]. In physical modeling it is used particularly for multidimensional mesh structures [7] but also for example in one-dimensional string modeling [9].

In the most commonly used way to discretize the wave equation by finite differences the partial derivatives

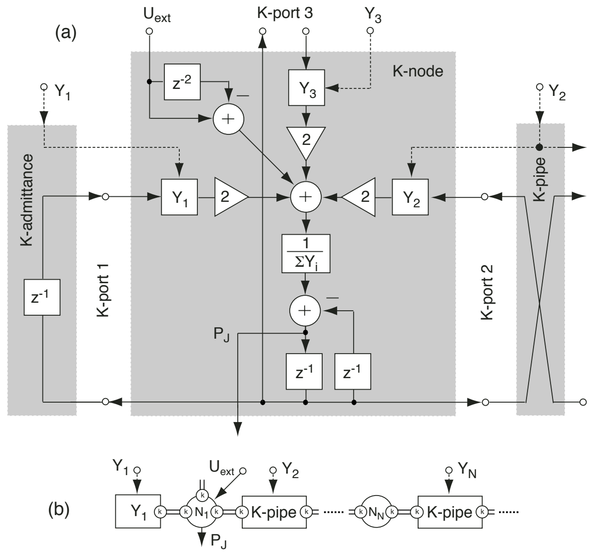

Figure 5: *Top*: An acoustic three-port parallel connection of FDTD type, corresponding to the DWG in Fig. 2. *Bottom*: Abstraction as a blockwise description of a 1-D FDTD waveguide.

in Eq. (1) are approximated by centered differences. Second-order partial derivatives are approximated by

$$y_{xx} \approx (y_{x+\Delta x,t} - 2y_{x,t} + y_{x-\Delta x,t})/\Delta x^2 \qquad (9)$$

$$y_{tt} \approx (y_{x,t+\Delta t} - 2y_{x,t} + y_{x,t-\Delta t})/\Delta t^2. \qquad (10)$$

where the notation $y_{x,t}$ is used instead of $y(x,t)$. By selecting $\Delta t = \Delta x/c$, and using index notation $k = x/\Delta x$ and $n = t/\Delta t$, Eqs. (9) and (10) result in

$$y_{k,n+1} = y_{k-1,n} + y_{k+1,n} - y_{k,n-1} \qquad (11)$$

From Eq. (11) we can see that a new sample $y_{k,n+1}$ at position k and time index $n+1$ is computed as the sum of its neighboring position values minus the value at the position itself one sample period earlier. Since $y_{k,n+1}$ is a physical variable, not a wave variable, we will classify the finite difference models as *K-models*, referring to Kirchhoff type of physical variables.

In [10] an extension of the scheme is derived that allows FDTD structures with arbitrary connection admittances to be formed, see Fig. 5, in a way resembling the DWG diagram in Fig. 2. A generalized recursion formulation for the structure (without excitation U_{ext}) becomes

$$P_{\text{J},n+1} = \frac{2}{Y_{\text{tot}}} \sum_{i=0}^{N-1} Y_i P_{i,n} - P_{\text{J},n-1} \qquad (12)$$

It is also shown in [10] that volume velocity excitation U_{ext} must be fed through a simple FIR filter $H(z) = 1 - z^{-2}$, so that the zeros of this filter cancel out the poles at DC and Nyquist frequency inherent in the DFTD structure due to the two-unit-delay feedback as shown in Fig. 5. The equivalence of the DWG in Fig. 2 and the FDTD structure in Fig. 5 has been proven in [10]. Details of their relations will be discussed in Section 3.1.

3 Relation of K- vs. W-modeling

Physics-based modeling is typically done using only one or a few of the paradigms presented above[1]. Researchers have their favorite method, and combinations of different approaches are rarely used to optimize the overall model.

As mentiond above, modeling paradigms can be categorized to *K-modeling* with Kirchhoff quantities and and *W-modeling* with wave quantities. Digital waveguides and wave digital filters are inherently based on W-modeling, while the FDTD formulation is based on K-modeling with a single K-variable (pressure in Fig. 5). Analog systems and circuits are typically best described by dual K-variables, such as voltage and current, force and velocity, pressure and volume velocity, etc.

3.1 Comparison of DWG vs. FDTD models

In [10] a careful analysis of DWGs and FDTDs (see Figs. 2 and 5) has been presented by proving their functional equivalence in processing K-variables related to these structures. The similarity is obvious when looking at the 'scattering substructure', i.e., summing of signals through admittances and inverse of the sum of the admittances. While in DWGs the delays in a waveguide structure are between scattering junctions, in the FDTD case the delays are implemented within the node itself by two unit delays. From this it is obvious that in two- or three-dimensional mesh-like structures FDTDs have the advantage of needing only two delays (memory positions) needed per node, while for the DWG mesh two delays are needed for each connection between junctions.

Among other differences between the DWG and FDTD structure are the lack of numerical robustness at DC and Nyquist frequencies in the FDTD structures as well as their inherent integrating property [10, 11].

3.2 DWG vs. FDTD compatibility

In [10] a method is given to construct mixed models by combining FDTD and DWG elements (thus also WDFs) through a K-to-W converter, a two-port element shown in Fig. 6. The left-hand side shows an FDTD junction and the right-hand side a DWG junction. The KW-converter maps the K-variable terminals of an FDTD junction to a wave port of a DWG junction and the other way around. The KW-converter allows for constructing hybrid models consisting of submodels designed either by K- or W-

[1]Other main modeling paradigms, not discussed here, are based on modal decomposition (modal synthesis), mass-spring models, and source-filter modeling.

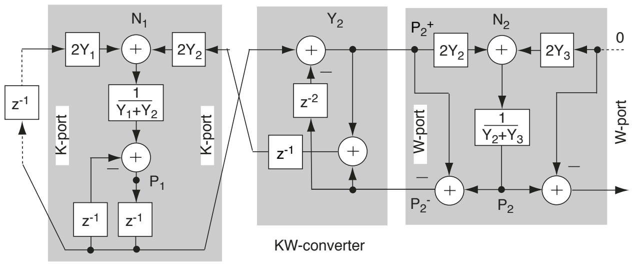

Figure 6: KW-converter for mixed modeling with FDTDs (left-hand side) and DWGs (right-hand side).

quantities. Thus DWGs, WDFs, and FDTDs are made compatible through this conversion.

3.3 Mixed FDTD + DWG 2-D mesh

Figure 7 illustrates a part of a 2-D mixed model structure that is based on a rectangular FDTD waveguide mesh for efficient and memory-saving computation but using DWG elements at boundaries. Such a model could be for example a drum membrane or in a 3-D case a room enclosed by walls. When there is need to attach W-type termination admittances to the model or to vary the propagation delays within the system, a change from K-elements to W-elements through converters is a useful property. This allows also using fractional delays and and other common DWG elements along with FDTD mesh structures.

In Fig. 7 the elements denoted by *kp* are K-type pipes (delay-free connections) between K-type nodes. Elements *kw* are K-to-W converters and elements *wl* are W-lines (delay-lines), where the arrows indicate that they are controllable fractional delays. Elements *yt* are terminating admittances. Practical modeling software for room acoustics, based on this principle, is reported in [12].

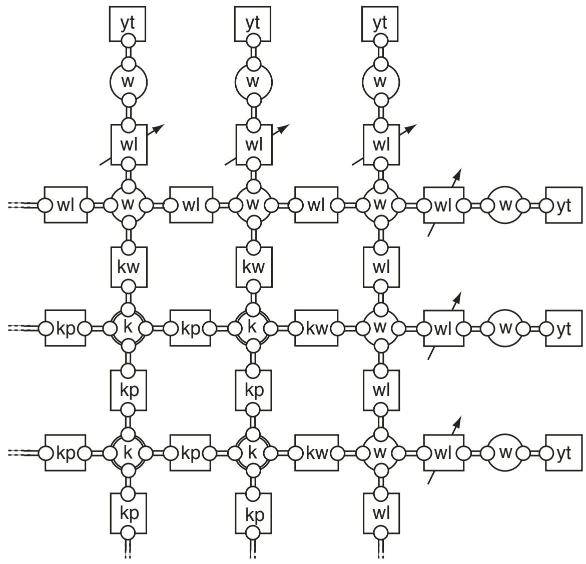

Figure 7: Part of a 2-D waveguide mesh composed of (a) K-type FDTD elements (left bottom): K-pipes (*kp*) and K-nodes (*k*), (b) W-type DWG elements (top and right): delay-controllable W-lines (*wl*), W-nodes (*w*), terminating admittances (*yt*), and (c) converter elements (*kw*) to connect K- and W-type elements into a mixed model.

4 Recent developments

Physics-based models using the paradigms discussed above are developed actively at the Helsinki University of Technology, Laboratory of Acoustics and Audio Signal Processing. One of the recent research topics, time-varying and nonlinear string models, is discussed below, and a larger scale modeling case, the Virtual Air Guitar, is mentioned briefly.

4.1 Energy-preserving string models

One of the most fundamental laws in physics is the preservation of total energy in a system. While true physical modeling techniques, such WDFs, guarantee proper energetic behavior in LTI cases, time varying models need special care to be taken to duplicate true physical behavior.

A recent reaserch topic of this kinds is modeling of strings of variable length [13]. For example when playing slide guitar with a relatively massive object sliging on the string, the energy of a lossless string should remain approximately constant. This does not happen for example in the Extended Karplus-Strong string loop depicted in Fig. 8(a) when fractional delay interpolation of the varying string length is computed by Lagrange interpolation, as shown in Fig. 8(b). According to notations in Fig. 8(c) the amount of energy ΔE is lost when the string is shortened and regained when the string is lengthened back:

$$\Delta E = E(x_1, x_2) = \int_{x_1}^{x_2} \frac{p^2(x)}{Z_0} dx \qquad (13)$$

where p is wave variable and Z_0 is wave impedance. When the length is changed by $\Delta x = x_2 - x_1$, Δx much smaller than wavelength and unit delay, a corrected value p_c can be used as interpolator output

$$\frac{p_\mathrm{c}^2}{Z_0} = \frac{p^2}{Z_0} - \Delta E = \frac{p^2}{Z_0} - \Delta x \frac{p^2}{Z_0} \qquad (14)$$

By sovlvng for p_C we obtain

$$p_\mathrm{c} = \sqrt{1 - \Delta x}\; p = g_\mathrm{c}\, p \approx (1 - \frac{\Delta x}{2}) p = g_\mathrm{ca}\, p \qquad (15)$$

Figure 9 shows a comparison of string models with and without energy compensation, demonstrating the desired behavior after compensation.

Another way of simulating a variable length energy-preserving string is by constructing the delay lines needed

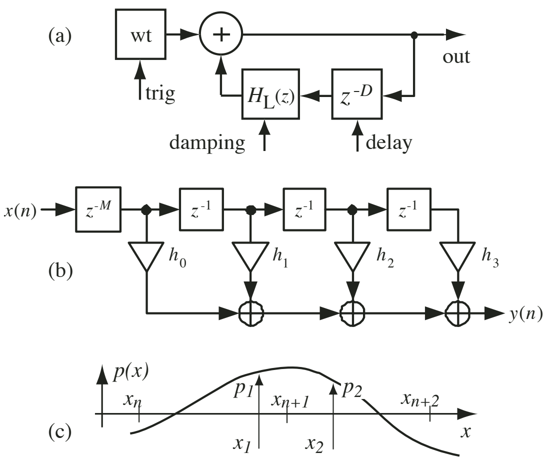

Figure 8: (a) Single-delay loop string model, (b) Lagrange interpolator for varying string length, and (c) definition of length interpolation for single sample interval.

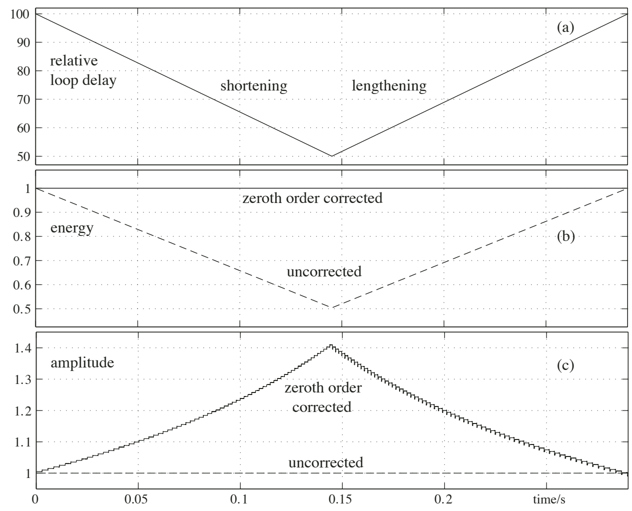

Figure 9: Evaluation of time-varying loop delay filter with (solid line) and without (dashed line) energy compensation: (a) sliding of loop delay in Fig. 8(a) from 100 % (128 taps) to 50 % (string shortening) and back to 100 %, (b) energy level, and (c) signal amplitude level.

as variable allpass filters that are designed through power-normalized WDF one-ports [13].

4.2 Virtual Air Guitar

As an example of playable virtual instrument we have developed a Virtual Air Guitar, described in detail in [14]. It consists of control sticks for pitch and plucking control, a synthetic electric guitar model for playing slide guitar 'power chords' (intervals of fifths) through simulated tube amplifier distortion and loudspeaker cabinet emulation. Time-varying models are needed for the slide guitar realization, and strongly nonlinear behavior is needed for tube amplifier overdriving. In both cases, but particularly for the distortion modeling, aliasing is a problem. Oversampling is a necessity in the distortion modeling in order to avoid audible artefacts.

Acknowledgments

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