Beamforming design for minimizing the signal power estimation error

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Beamforming

Honey, I shrunk the Capon's beamformer

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Sensor Array Signal Processing

- In **sensor array signal processing** [KV96, VT02] [SM05, Chapt. 6] a group of sensors located at distinct spatial locations is employed.
- A propagating wavefield is measured with an array of M sensors and a M-channel observation is formed.
- Spatial sampling may be non-uniform and arrays may be multidimensional (linear, spherical, rectangular).
- Applications: wireless comms, radar, channel sounding and propagation modelling, electronic warfare, EEG/MEG, sonar, seismology and astronomy.

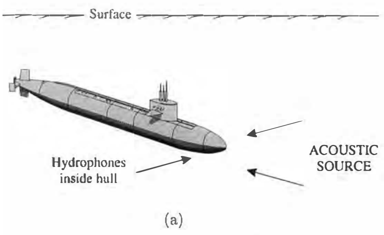


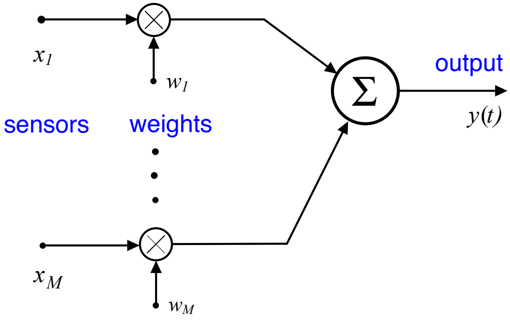
Fig: submarine mounted hydrophone arrays. Fig. from [VT02].

Beamforming

- A **spatial filter** linearly combines the M sensor signals at time t
- The spatial filter output is

$$\hat{s}(t) = \mathbf{w}^{\mathsf{H}}\mathbf{x}(t)$$
$$= w_1^* x_1(t) + \ldots + w_M^* x_M(t)$$

  where the input to the filter is $\mathbf{x}(t) = (x_1(t), \ldots, x_M(t))^\top$.
- Such filtering is called **beamforming**.



Pick the direction of interest θ s.t. $\hat{s}(t) = \mathbf{w}^{\mathsf{H}}\mathbf{x}(t)$ contain mostly the signal $s(t)$ from θ.

Nomenclature

Definitions:

- $\mathbf{w} \in \mathbb{C}^M$:= **beamformer weight vector**
- $\hat{s}(t) = \mathbf{w}^{\mathsf{H}}\mathbf{x}(t)$:= **beamformer output** or **signal waveform estimate**
- $\mathbf{\Sigma} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}(t)^{\mathsf{H}}]$:= **array covariance matrix**
- $\mathbf{x}(1), \ldots, \mathbf{x}(T) \in \mathbb{C}^M$:= **snapshots** from array at T distinct time instances
- $\mathbf{a}(\theta) \in \mathbb{C}^M$:= **steering vector** or **array response** towards direction θ

Goal: enhance the **signal-of-interest (SOI)** from direction θ while attenuate undesired signals (interferers) from other directions

Spatial specrum

- The beamformer **output power**:

$$P(\theta) = \mathbb{E}[|\mathbf{w}^{\mathsf{H}}\mathbf{x}(t)|^2] = \mathbf{w}^{\mathsf{H}}\boldsymbol{\Sigma}\mathbf{w}$$

  provides an indication of the amount of energy coming from direction θ.
- Dependency on θ is through weight $\mathbf{w} = \mathbf{w}(\theta)$, that is optimized for given θ
- Observed (empirical) power:

$$\hat{P}(\theta) = \frac{1}{T}\sum_{t=1}^{T} |\mathbf{w}^{\mathsf{H}}\mathbf{x}(t)|^2 = \mathbf{w}^{\mathsf{H}} \underbrace{\left(\frac{1}{T}\sum_{t=1}^{T} \mathbf{x}(t)\mathbf{x}(t)^{\mathsf{H}}\right)}_{=\hat{\boldsymbol{\Sigma}}} \mathbf{w} = \mathbf{w}^{\mathsf{H}}\hat{\boldsymbol{\Sigma}}\mathbf{w}$$

  where $\hat{\boldsymbol{\Sigma}}$ is the **sample covariance matrix**
- Graph of $\hat{P}(\theta)$ w.r.t. look direction θ is called the **spatial spectrum**.

Capon Beamformer

- Capon [Cap69]: When multiple sources are present, the power at direction θ includes not only the source's power but also contributions from interfering signals.
- This property limits the resolution of conventional beamformer.
- Distortionless response constraint: $\mathbf{w}^{\mathsf{H}}\mathbf{a}(\theta) = 1$.
- Objective: choose $\mathbf{w}$ to minimize the power $\mathbb{E}[|\mathbf{w}^{\mathsf{H}}\mathbf{x}(t)|] = \mathbf{w}^{\mathsf{H}}\boldsymbol{\Sigma}\mathbf{w}$

$$\min_{\mathbf{w}} \mathbf{w}^{\mathsf{H}}\boldsymbol{\Sigma}\mathbf{w} \quad \text{subject to } \mathbf{w}^{\mathsf{H}}\mathbf{a}(\theta) = 1$$

Solution:

$$\mathbf{w}_{\mathsf{Cap}} = \gamma_{\mathsf{Cap}}\boldsymbol{\Sigma}^{-1}\mathbf{a}(\theta)$$

$$\gamma_{\mathsf{Cap}} = \frac{1}{\mathbf{a}(\theta)^{\mathsf{H}}\boldsymbol{\Sigma}^{-1}\mathbf{a}(\theta)}$$

Scalar term is the output power at θ:

$$P(\theta) = \mathbb{E}[|\mathbf{w}_{\mathsf{Cap}}\mathbf{x}(t)|^2] = \gamma_{\mathsf{Cap}}$$

Other names: minimum power distortionless constraint (MPDR) beamformer, or MVDR

How good is Capon's beamformer in estimating the power?

- $K = 4$ sources at directions

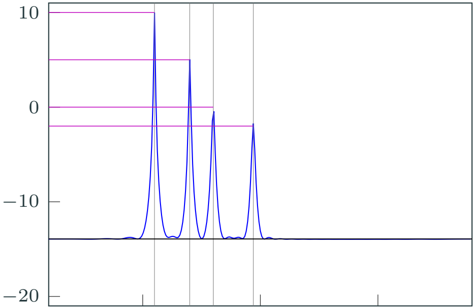
$$\theta_k = -45.02^o, -30.02^o, -20.02^o, -3^o$$

- ULA of $M = 25$ elements and $\lambda/2$ spacing.
- source signals $s_k(t)$ are random 8-PSK modulated with fixed amplitudes:

$$\gamma_k = |s_k(t)|^2 = 10, 5, 0, -2 \text{ dB}$$

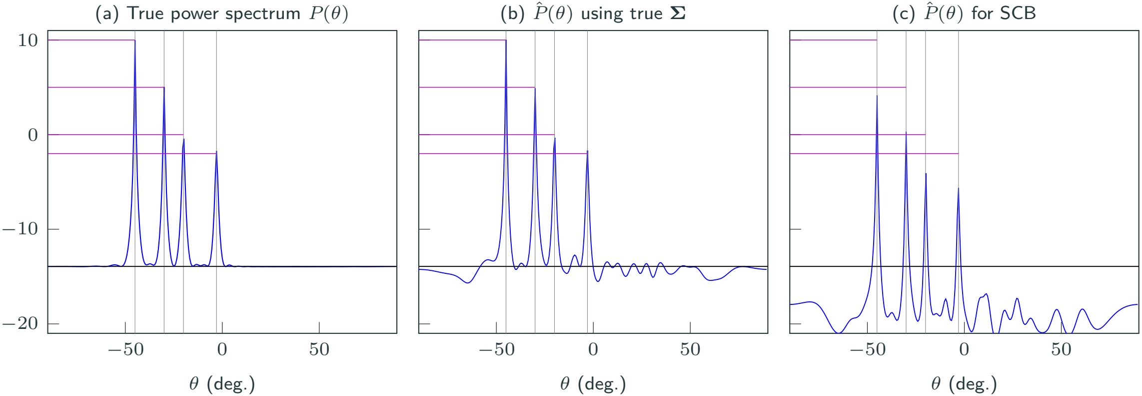
- Noise is Gaussian of unit variance $\sigma^2 = 1$
- $T = 35$ snapshots.

(a) True power spectrum $P(\theta)$



Pretty good! All powers of the 4 signals (horizontal lines) are accurately estimated by the power spectrum

But not in practise!



Standard Capon Beamformer (SCB) $\hat{\mathbf{w}}_{\mathsf{Cap}}$: use $\hat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}(t)\mathbf{x}(t)^{\mathsf{H}}$ in place of unknown $\boldsymbol{\Sigma}$

Oops! powers are completely off (***biased!***), but signal directions are accurately found by SCB

PAB and MMSE beamformers

The phased array beamformer (PAB)

- Distortionless constraint $\mathbf{w}^{\mathsf{H}}\mathbf{a}(\theta) = 1$.
- Objective: maximize the output SNR
- Assumption: single source signal in white noise:

$$\mathbf{x}(t) = \mathbf{a}(\theta)s(t) + \mathbf{e}(t)$$

where $\mathbf{Q} = \mathbb{E}[\mathbf{e}(t)\mathbf{e}(t)^{\mathsf{H}}] = \sigma^2\mathbf{I}$.

- Solution:

$$\mathbf{w}_{\text{PAB}} = \frac{\mathbf{a}(\theta)}{\|\mathbf{a}(\theta)\|^2}$$

The MMSE beamformer

- Signal estimate $\hat{s}(t) = \mathbf{w}^{\mathsf{H}}\mathbf{x}(t)$
- Signal estimation mean squared error (SE-MSE):

$$\mathbb{E}[|s(t) - \hat{s}(t)|^2]$$

- Objective: minimize SE-MSE
- Solution:

$$\mathbf{w}_{\text{MMSE}} = \gamma\mathbf{\Sigma}^{-1}\mathbf{a}(\theta),$$

where $\gamma = \mathbb{E}[|s(t)|^2]$ the power of SOI

Power estimation bias

- The array covariance matrix has the form:

$$\boldsymbol{\Sigma} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}(t)^{\mathsf{H}}] = \gamma \mathbf{a}(\theta)\mathbf{a}(\theta)^{\mathsf{H}} + \mathbf{Q},$$

where $\mathbf{Q}$ is the ***interference-plus-noise covariance matrix*** (INCM).

- Biases of power estimators $\hat{\gamma} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{w}^{\mathsf{H}}\mathbf{x}(t)$:

$$\mathrm{B}(\hat{\gamma}_{\mathsf{PAB}}) = \mathbb{E}[|\mathbf{w}_{\mathsf{PAB}}^{\mathsf{H}}\mathbf{x}(t)|^2] - \gamma = \frac{\mathbf{a}(\theta)^{\mathsf{H}}\mathbf{Q}\mathbf{a}(\theta)}{\|\mathbf{a}(\theta)\|^4} > 0$$

$$\mathrm{B}(\hat{\gamma}_{\mathsf{Cap}}) = \mathbb{E}[|\mathbf{w}_{\mathsf{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^2] - \gamma = \frac{1}{\mathbf{a}^{\mathsf{H}}(\theta)\mathbf{Q}^{-1}\mathbf{a}(\theta)} > 0$$

$$\mathrm{B}(\hat{\gamma}_{\mathsf{MMSE}}) = \mathbb{E}[|\mathbf{w}_{\mathsf{MMSE}}^{\mathsf{H}}\mathbf{x}(t)|^2] - \gamma = \gamma\left(\frac{\gamma}{\gamma_{\mathsf{Cap}}} - 1\right) < 0$$

- Hence, we have $\mathbb{E}[|\mathbf{w}_{\mathsf{MMSE}}^{\mathsf{H}}\mathbf{x}(t)|^2] = \gamma_{\mathsf{MMSE}} < \gamma < \gamma_{\mathsf{Cap}} = \mathbb{E}[|\mathbf{w}_{\mathsf{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^2]$.

- Capon overshoots, MMSE undershoots $\Rightarrow$ Let's ***shrink*** the Capon's beamformer! 

Example

Set-up:

- Uniform Linear Array (ULA) with $M = 25$ antennas, $\lambda/2$ sensor spacing,
- Steering vector:

$$\mathbf{a}(\theta) \triangleq (1, e^{-\jmath \cdot 1 \cdot \pi \sin\theta}, \ldots, e^{-\jmath \cdot (M-1) \cdot \pi \sin\theta})^\top,$$

  for $\theta \in \Theta = [-\pi/2, \pi/2)$ in radians
- Four indep. complex Gaussian sources:
  - SOI has DOA -45.02°.
  - Interfering sources: -30.02°, -20.02°, -3° of 2, 4, and 6 dB lower power than SOI, respectively.
- Noise is white Gaussian with unit variance.

Metrics:

- Signal estimation NMSE:

$$\text{SE-NMSE}_T = \frac{\sum_{t=1}^{T} |\hat{s}(t) - s(t)|^2}{\sum_{t=1}^{T} |s(t)|^2}$$

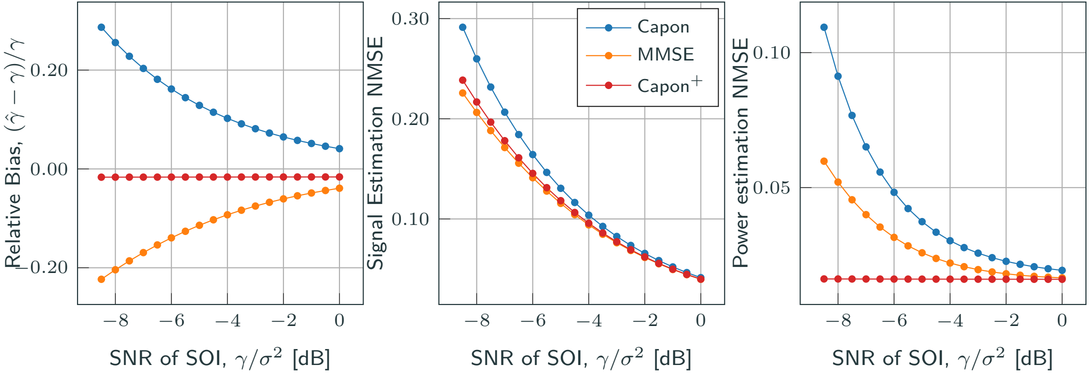
- Signal power estimation NMSE:

$$\text{SP-NMSE}_T = \frac{(\hat{\gamma} - \gamma)^2}{\gamma^2}.$$

where

$$\hat{\gamma} = \frac{1}{T}\sum_{t=1}^{T} |\hat{s}(t)| = \frac{1}{T}\sum_{t=1}^{T} |\mathbf{w}^{\mathsf{H}}\mathbf{x}(t)|^2$$

Example



Capon$^+$ is the proposed method, to be introduced next.

Shrinking the Capon beamformer

- Consider a shrinkage beamformer:

$$\mathbf{w}_\beta = \beta \mathbf{w}_{\mathsf{Cap}}, \qquad \beta > 0.$$

  - $\beta = 1 :=$ Capon
  - $\beta = \gamma/\gamma_{\mathsf{Cap}}$:=MMSE

- Power estimate:

$$\hat{\gamma}_{\mathsf{Cap}^+} = \frac{1}{T}\sum_{t=1}^{T} |\mathbf{w}_\beta^{\mathsf{H}} \mathbf{x}(t)|^2 = \alpha \hat{\gamma}_{\mathsf{Cap}},$$

with $\alpha = \beta^2$.

Objective: choose α that minimizes the $\mathsf{MSE}(\hat{\gamma}) = \mathsf{var}(\hat{\gamma}) + [\mathsf{B}(\hat{\gamma})]^2$

Shrinking the Capon beamformer

- Consider a shrinkage beamformer:

$$\mathbf{w}_\beta = \beta \mathbf{w}_{\mathsf{Cap}}, \qquad \beta > 0.$$

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  - $\beta = \gamma/\gamma_{\mathsf{Cap}} :=$MMSE *negative bias*

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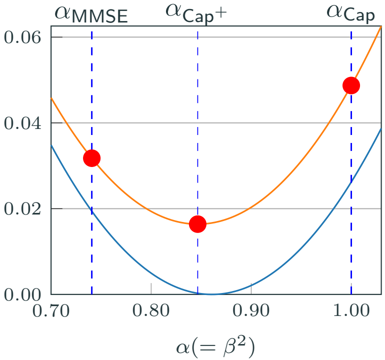

orange line: MSE, blue: bias^2

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Positive for Capon, negative for MMSE

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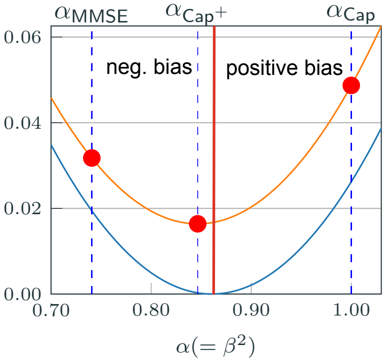

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Positive for Capon, negative for MMSE

Shrinking the Capon beamformer

Theorem

The value of α that minimizes the MSE is

$$\alpha_{\mathsf{Cap}^+} = \frac{T\gamma_{\mathsf{Cap}}\gamma}{\mathbb{E}[|\mathbf{w}_{\mathsf{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^4] + (T-1)\gamma_{\mathsf{Cap}}^2},$$

where $\gamma_{\mathsf{Cap}} = \mathbb{E}[|\mathbf{w}_{\mathsf{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^2]$.

The minimum MSE is

$$\mathsf{MSE}_{\min} = \mathbb{E}[(\alpha_{\mathsf{Cap}^+}\hat{\gamma}_{\mathsf{Cap}} - \gamma)^2] = \gamma^2 \frac{\mathsf{var}(\hat{\gamma}_{\mathsf{Cap}})}{\mathbb{E}[\hat{\gamma}_{\mathsf{Cap}}^2]},$$

where $\mathsf{var}(\hat{\gamma}_{\mathsf{Cap}}) = (\mathbb{E}[|\mathbf{w}_{\mathsf{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^4] - \gamma_{\mathsf{Cap}}^2)/T$.

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Some notes

- Minimum MSE does not depend on SNR

Shrinking the Capon beamformer

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Some notes

- Minimum MSE does not depend on SNR
- We can express α_{Cap^+} as

$$\alpha_{\mathsf{Cap}^+} = \tau \cdot \underbrace{\frac{\gamma}{\gamma_{\mathsf{Cap}}}}_{<1}, \quad \tau = \frac{T}{\mathsf{kurt}(\hat{s}(t)) + T + 1}$$

and

$$\mathsf{kurt}(\hat{s}(t)) = \frac{\mathbb{E}[|\hat{s}(t)|^4]}{(\mathbb{E}[|\hat{s}(t)|^2])^2} - 2 := \mathsf{kurtosis}$$

Shrinking the Capon beamformer

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where $\gamma_{\text{Cap}} = \mathbb{E}[|\mathbf{w}_{\text{Cap}}^{\text{H}}\mathbf{x}(t)|^2]$.

The minimum MSE is

$$\text{MSE}_{\min} = \mathbb{E}[(\alpha_{\text{Cap}^+}\hat{\gamma}_{\text{Cap}} - \gamma)^2] = \gamma^2 \frac{\text{var}(\hat{\gamma}_{\text{Cap}})}{\mathbb{E}[\hat{\gamma}_{\text{Cap}}^2]},$$

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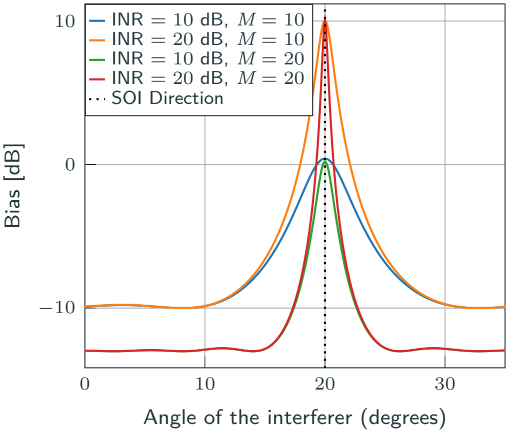
- Note: $\text{kurt}(s) \geq 1$ and $\text{kurt}(s) = -1$ if s is random constant modulus signal.

On the bias of the Capon$^+$ beamformer

- Capon and MMSE beamformers are not asymptotically ($T \to \infty$) unbiased either.
- Namely: $\mathrm{B}(\hat{\gamma}_{\mathrm{Cap}}) \to 0$ only if $M \to \infty$ (and INCM $\mathbf{Q}$ is "well-behaved") or $\sigma^2 \to 0$.
- The bias depends on array size M, and proximity of SOI to interferers.
- Figure demonstrates the case of a single interferer and SOI at 20^o



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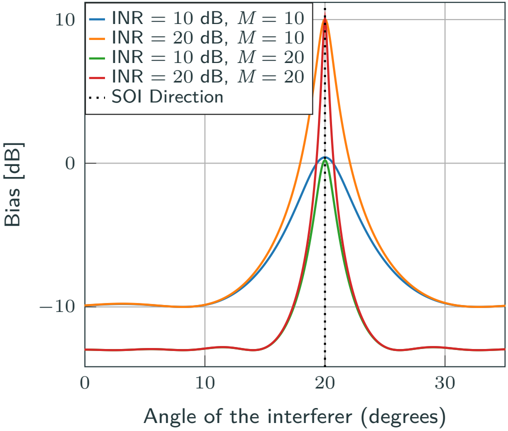
! $\hat{\gamma}_{\mathsf{Cap}^+}$ is asymptotically unbiased, and bias for finite samples often negligible



Simulation study

- Capon$^+$ beamformer has good performance, as demonstrated earlier.
- BUT: it does not guarantee that its *adaptive version*, i.e., when based on sample-based estimates $\{\mathbf{x}(t)\}_{t=1}^{T}$, will have good performance.
- We consider some examples, where adaptive Capon$^+$ beamformer can be used.

Set-up:

- ULA with $\lambda/2$ spacing, $M = 25$ antennas.
- 8-PSK modulated random signals with fixed constant squared amplitude, $\gamma_k = |s_k(t)|^2$, $k = 1, \ldots, 4$.
- DOAs of the interfering sources are sampled uniformly from the set $[-90, \theta - 3] \cup [\theta + 3, 90]$ degrees, where θ is the DOA of the SOI.

Scenario A

- INCM $\mathbf{Q}$ is ***known*** $\Rightarrow$ $\mathbf{w}_{\text{Cap}}$ is known exactly.
- γ is ***unknown***, and needs to be estimated

$\Rightarrow$ we used debiased Capon power estimate:

$$\hat{\gamma}_{\text{deb}} = \max(\hat{\gamma}_{\text{Cap}} - \underbrace{(\mathbf{a}(\theta)^{\mathsf{H}}\mathbf{Q}^{-1}\mathbf{a}(\theta))^{-1}}_{\text{bias}}, 0),$$

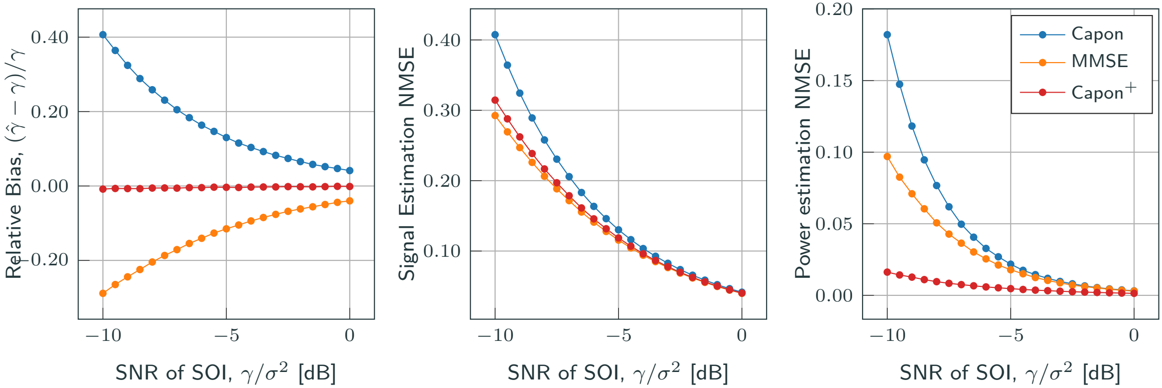
where $\hat{\gamma}_{\text{Cap}} = \frac{1}{T}\sum_{t=1}^{T} |\mathbf{w}_{\text{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^2$.

- Adaptive shrinkage for Capon$^+$ beamformer:

$$\hat{\alpha}_{\text{Cap}^+} = \frac{T\hat{\gamma}_{\text{Cap}}\hat{\gamma}_{\text{deb}}}{\frac{1}{T}\sum_{t=1}^{T} |\mathbf{w}_{\text{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^4 + (T-1)\hat{\gamma}_{\text{Cap}}^2}.$$

- Capon$^+$: $\mathbf{w}_{\text{Cap}^+} = \sqrt{\hat{\alpha}_{\text{Cap}^+}}\mathbf{w}_{\text{Cap}}$.
- MMSE beamformer $\mathbf{w}_{\text{MMSE}}$ known up scaling γ: we use estimate $\hat{\gamma}_{\text{deb}}$.

Results for scenario A ($T = 60$)



Scenario B

- INCM $\mathbf{Q}$ is ***unknown***, γ is **known**

$\Rightarrow$ need to use adaptive Capon beamformer, where $\mathbf{\Sigma}$ estimated by the SCM $\hat{\mathbf{\Sigma}}$:

$$\hat{\mathbf{w}}_{\text{Cap}} = \hat{\hat{\gamma}}_{\text{Cap}} \hat{\mathbf{\Sigma}}^{-1} \mathbf{a}(\theta), \quad \hat{\hat{\gamma}}_{\text{Cap}} = (\mathbf{a}(\theta)^{\mathsf{H}} \hat{\mathbf{\Sigma}}^{-1} \mathbf{a}(\theta))^{-1},$$

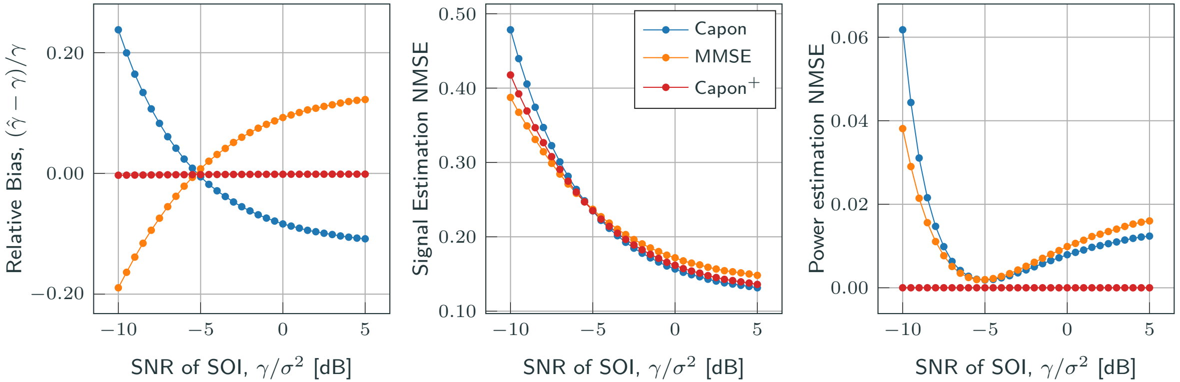
where $\hat{\mathbf{\Sigma}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{x}(t)\mathbf{x}(t)^{\mathsf{H}}$.

- Shrinkage for Capon$^+$ beamformer:

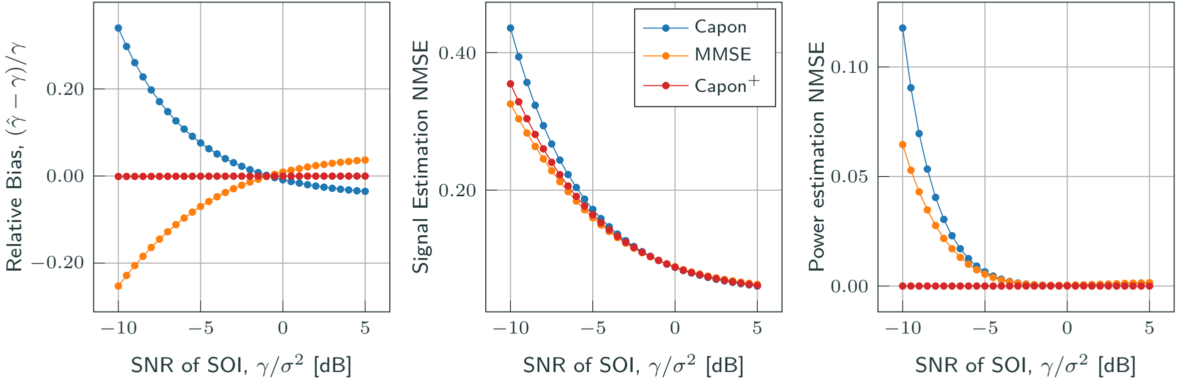
$$\hat{\alpha}_{\text{Cap}^+} = \frac{T \hat{\hat{\gamma}}_{\text{Cap}} \gamma}{\frac{1}{T} \sum_{t=1}^{T} |\hat{\mathbf{w}}_{\text{Cap}}^{\mathsf{H}} \mathbf{x}(t)|^4 + (T-1)\hat{\hat{\gamma}}_{\text{Cap}}^2}.$$

- Capon$^+$: $\hat{\mathbf{w}}_{\text{Cap}^+} = \sqrt{\hat{\alpha}_{\text{Cap}^+}} \hat{\mathbf{w}}_{\text{Cap}}$.
- MMSE: $\hat{\mathbf{w}}_{\text{MMSE}} = \gamma \hat{\mathbf{\Sigma}}^{-1} \mathbf{a}(\theta)$

Results for scenario B ($T = 200$)



Results for scenario B ($T = 500$)



Conclusions

- We *shrunk the Capon's beamformer*
- Capon's beamformer has overestimation bias, while MMSE beamformer has underestimation bias. Neither are asymptotically unbiased, unlike the Capon$^+$.
- the Capon$^+$ provides an essentially unbiased signal power estimate across all SNRs and matches MMSE beamformer in signal estimation NMSE.
- Capon$^+$ strikes the best balance between signal power and waveform estimation while exhibiting minimal bias.

Related works

- [JS99] had finite sample bias study of the adaptive Capon's beamformer and showed that it tends to *underestimate* the power in small samples.
- [ENLR07] proposed robust enhancements of the MMSE beamformer (**??**) taking into account imperfect knowledge of the SOI signal power level.
- A linear combination of adaptive Capon's beamformer and the phased array beamformer was considered in [SN14] to optimize the MMSE objective.

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