APPROACH TO BEAMFORMING MINIMIZING THE SIGNAL POWER ESTIMATION ERROR

Esa Ollila
Aalto University, Finland

Xavier Mestre
CTTC, Spain

Elias Raninen
Nokia Bell Labs, Finland

INTRODUCTION

- Capon and MMSE beamformers overestimate or underestimate the signal power, respectively; they are not asymptotically unbiased either.
- We propose Capon$^+$ beamformer: scaled Capon beamformer with scaling factor chosen to minimize the MSE of the signal power estimate.
- It provides a superior balance in both signal power and waveform estimation while exhibiting minimal bias, that tends to 0 as $T \to \infty$.

RECEIVED SIGNAL

- Consider an array of M sensors and data (snapshots):

$$\mathbf{x}(t) = s(t)\mathbf{a} + \mathbf{e}(t), \; t = 1, \ldots, T,$$

  - $s(t) \in \mathbb{C}$ is the signal waveform of the *signal of interest* (SOI).
  - $\mathbf{a} \in \mathbb{C}^M$ is the steering vector for the SOI.
  - $\mathbf{e}(t) \in \mathbb{C}^M$ is a random vector of interference and noise.
- The array covariance matrix has the form:

$$\boldsymbol{\Sigma} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}(t)^{\mathsf{H}}] = \gamma \mathbf{a}\mathbf{a}^{\mathsf{H}} + \mathbf{Q},$$

  - $\gamma = \mathbb{E}[|s(t)|^2]$ is the SOI power.
  - $\mathbf{Q} = \mathbb{E}[\mathbf{e}(t)\mathbf{e}(t)^{\mathsf{H}}]$ is the *interference-plus-noise covariance matrix* (INCM).

BEAMFORMING

- Signal waveform estimate with given beamformer $\mathbf{w} \in \mathbb{C}^M$:

$$\hat{s}(t) = \mathbf{w}^{\mathsf{H}}\mathbf{x}(t).$$

- Estimated and expected beamformer power for fixed $\mathbf{w}$:

$$\hat{\gamma} = \frac{1}{T}\sum_{t=1}^{T} |\hat{s}(t)|^2, \qquad \mathbb{E}[\hat{\gamma}] = \mathbb{E}[|\hat{s}(t)|^2] = \mathbf{w}^{\mathsf{H}}\boldsymbol{\Sigma}\mathbf{w}.$$

- The *Capon beamformer* [1] [2, Sec. 6.2.4]

$$\min_{\mathbf{w}} \mathbf{w}^{\mathsf{H}}\boldsymbol{\Sigma}\mathbf{w} \quad \text{subject to } \mathbf{w}^{\mathsf{H}}\mathbf{a} = 1.$$

$$\Rightarrow \mathbf{w}_{\text{Cap}} = \gamma_{\text{Cap}}\boldsymbol{\Sigma}^{-1}\mathbf{a} = \frac{\mathbf{Q}^{-1}\mathbf{a}}{\mathbf{a}^{\mathsf{H}}\mathbf{Q}^{-1}\mathbf{a}}, \quad \text{where} \quad \gamma_{\text{Cap}} = \frac{1}{\mathbf{a}^{\mathsf{H}}\boldsymbol{\Sigma}^{-1}\mathbf{a}}.$$

- The *MMSE beamformer* [2, Sec. 6.2.2]

$$\min_{\mathbf{w}} \left\{ \text{MSE}(\mathbf{w}) = \mathbb{E}[|s(t) - \mathbf{w}^{\mathsf{H}}\mathbf{x}(t)|^2] \right\}.$$

$$\Rightarrow \mathbf{w}_{\text{MMSE}} = \gamma\boldsymbol{\Sigma}^{-1}\mathbf{a} = \frac{\gamma}{\gamma_{\text{Cap}}}\mathbf{w}_{\text{Cap}}.$$

POWER ESTIMATION BIAS

- Power estimation bias of the Capon beamformer:

$$\begin{aligned}\mathsf{B}(\hat{\gamma}_{\text{Cap}}) &= \mathbf{w}_{\text{Cap}}^{\mathsf{H}}\boldsymbol{\Sigma}\mathbf{w}_{\text{Cap}} - \gamma = \mathbf{w}_{\text{Cap}}^{\mathsf{H}}(\gamma\mathbf{a}\mathbf{a}^{\mathsf{H}} + \mathbf{Q})\mathbf{w}_{\text{Cap}} - \gamma \\ &= \frac{1}{\mathbf{a}^{\mathsf{H}}\mathbf{Q}^{-1}\mathbf{a}} > 0.\end{aligned}$$

- Power estimation bias of the MMSE beamformer:

$$\mathsf{B}(\hat{\gamma}_{\text{MMSE}}) = \mathbf{w}_{\text{MMSE}}^{\mathsf{H}}\boldsymbol{\Sigma}\mathbf{w}_{\text{MMSE}} - \gamma = \gamma^2\mathbf{a}^{\mathsf{H}}\boldsymbol{\Sigma}^{-1}\mathbf{a} - \gamma = \gamma\left(\frac{\gamma}{\gamma_{\text{Cap}}} - 1\right) < 0.$$

- Hence, we have $\gamma_{\text{MMSE}} < \gamma < \gamma_{\text{Cap}}$.

CAPON$^+$ BEAMFORMER

- Consider a shrinkage beamformer:

$$\mathbf{w}_\beta = \beta\mathbf{w}_{\text{Cap}}, \qquad \beta > 0.$$

- Power estimate: $\hat{\gamma}_{\text{Cap}^+} = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T} |\mathbf{w}_\beta^{\mathsf{H}}\mathbf{x}(t)|^2 = \alpha\hat{\gamma}_{\text{Cap}}$, with $\alpha = \beta^2$.

Theorem 1. *The value of α that minimizes the MSE $\mathbb{E}[(\alpha\hat{\gamma}_{\text{Cap}} - \gamma)^2]$ is*

$$\alpha_o = \frac{T\gamma_{\text{Cap}}\gamma}{\mathbb{E}[|\mathbf{w}_{\text{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^4] + (T-1)\gamma_{\text{Cap}}^2}.$$

SIMULATIONS

- Uniform Linear Array (ULA) with $M = 25$ antennas and steering vector:

$$\mathbf{a}(\theta) \triangleq (1, e^{-\jmath \cdot 1 \cdot \frac{2\pi d}{\lambda}\sin\theta}, \ldots, e^{-\jmath \cdot (M-1) \cdot \frac{2\pi d}{\lambda}\sin\theta})^\top,$$

  - λ is the wavelength, d is the element spacing between the sensors (we assume $d = \lambda/2$).
  - $\theta \in \Theta = [-\pi/2, \pi/2)$ is the direction-of-arrival (DOA) of the SOI in radians.
- Four independent complex Gaussian sources:
  - SOI has DOA -45.02°.
  - Interfering sources have DOAs: -30.02°, -20.02°, -3° and signal powers 2, 4, and 6 dB lower than the power of the SOI, respectively.
  - Noise is white Gaussian with unit variance.
- All metrics are averaged over 15000 MC trials.

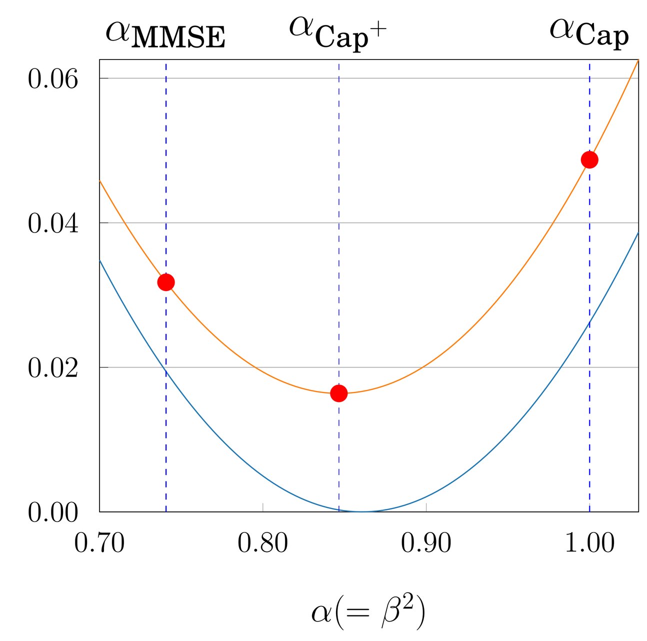


Fig. The NMSE (yellow) and the relative squared bias (blue) of $\hat{\gamma} = \alpha\hat{\gamma}_{\text{Cap}}$ as a function of α when SOI has -6 dB SNR. Beamformers $\mathbf{w}_{\text{MMSE}}$, $\mathbf{w}_{\text{Cap}}$ or $\mathbf{w}_{\text{Cap}^+}$ computed with known $\mathbf{Q}$ and γ. Number of snapshots is $T = 60$.

PERFORMANCE METRICS

- Relative bias: $(\hat{\gamma} - \gamma)/\gamma$.
- Signal estimation NMSE, SE-NMSE$_T = \sum_{t=1}^{T} |\hat{s}(t) - s(t)|^2 \big/ \sum_{t=1}^{T} |s(t)|^2$.
- Signal power estimation NMSE: SP-NMSE$_T = \dfrac{(\hat{\gamma} - \gamma)^2}{\gamma^2}$.

SCENARIO A, $T = 60$

- INCM $\mathbf{Q}$ is known, γ is unknown
- Debiased Capon power estimate:

$$\hat{\gamma}_{\text{deb}} = \max(\hat{\gamma}_{\text{Cap}} - (\mathbf{a}^{\mathsf{H}}\mathbf{Q}^{-1}\mathbf{a})^{-1}, 0),$$

  where $\hat{\gamma}_{\text{Cap}} = \frac{1}{T}\sum_{t=1}^{T} |\mathbf{w}_{\text{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^2$.
- Shrinkage for Capon$^+$ beamformer:

$$\hat{\alpha}_{\text{Cap}^+} = \frac{T\hat{\gamma}_{\text{Cap}}\hat{\gamma}_{\text{deb}}}{\frac{1}{T}\sum_{t=1}^{T} |\mathbf{w}_{\text{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^4 + (T-1)\hat{\gamma}_{\text{Cap}}^2}.$$

- Capon$^+$: $\mathbf{w}_{\text{Cap}^+} = \sqrt{\hat{\alpha}_{\text{Cap}^+}}\mathbf{w}_{\text{Cap}}$.

SCENARIO B, $T = 200$

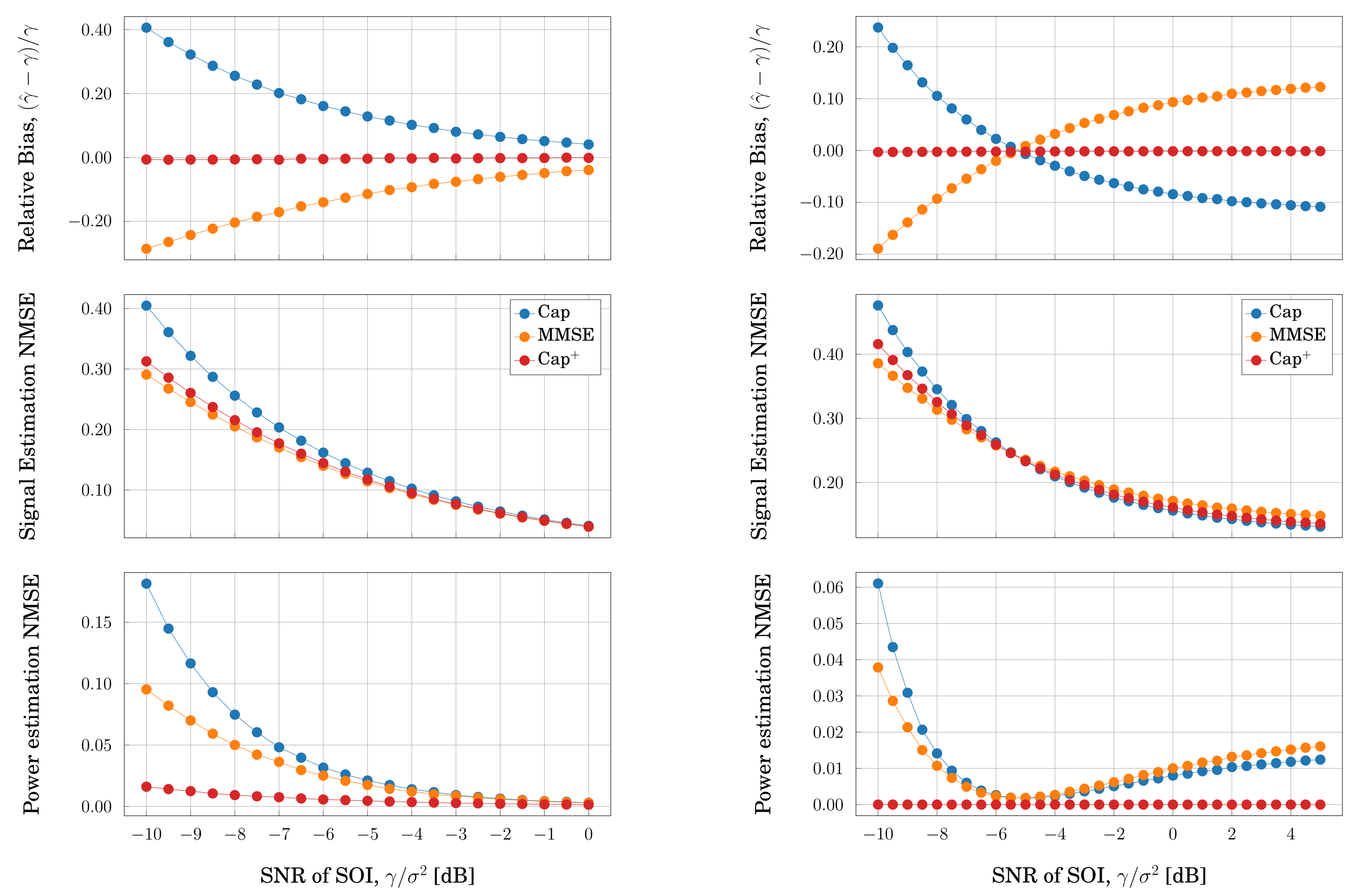
- $\mathbf{Q}$ is unknown, γ is known.
- Capon beamformer estimate:

$$\hat{\mathbf{w}}_{\text{Cap}} = \hat{\hat{\gamma}}_{\text{Cap}}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{a}, \quad \hat{\hat{\gamma}}_{\text{Cap}} = (\mathbf{a}^{\mathsf{H}}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{a})^{-1},$$

  where $\hat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}(t)\mathbf{x}(t)^{\mathsf{H}}$.
- Shrinkage for Capon$^+$ beamformer:

$$\hat{\alpha}_{\text{Cap}^+} = \frac{T\hat{\hat{\gamma}}_{\text{Cap}}\gamma}{\frac{1}{T}\sum_{t=1}^{T} |\hat{\mathbf{w}}_{\text{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^4 + (T-1)\hat{\hat{\gamma}}_{\text{Cap}}^2}.$$

- Capon$^+$: $\hat{\mathbf{w}}_{\text{Cap}^+} = \sqrt{\hat{\alpha}_{\text{Cap}^+}}\hat{\mathbf{w}}_{\text{Cap}}$.



CONCLUSION

- Capon$^+$ provides an essentially unbiased signal power estimate across all SNRs and matches MMSE beamformer in signal estimation NMSE.
- Capon$^+$ strikes the best balance between signal power and waveform estimation while exhibiting minimal bias.

REFERENCES

[1] Jack Capon, "High-resolution frequency-wavenumber spectrum analysis," *Proceedings of the IEEE*, vol. 57, no. 8, pp. 1408–1418, 1969.

[2] Harry L Van Trees, *Optimum array processing: Part IV of detection, estimation, and modulation theory*, John Wiley & Sons, 2002.

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